

The Mass Shell of the Nelson Model without Cut-Offs

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January 19, 2013

Abstract

The massless Nelson model describes non-relativistic, spinless quantum particles interacting with a relativistic, massless, scalar quantum field. The interaction is linear in the field. We analyze its one particle sector. First, we construct the renormalized mass shell of the non-relativistic particle for an arbitrarily small infrared cut-off that turns off the interaction with the low energy modes of the field. No ultraviolet cut-off is imposed. Second, we implement a suitable Bogolyubov transformation of the Hamiltonian in the infrared regime. This transformation depends on the total momentum of the system and is non-unitary as the infrared cut-off is removed. For the transformed Hamiltonian we construct the mass shell in the limit where both the ultraviolet and the infrared cut-off are removed. Our approach is constructive and leads to explicit expansion formulae which are amenable to rigorously control the S-matrix elements.

Keywords: Multiscale Perturbation Theory, Nelson Model, Renormalization, Ultraviolet Divergence, Infrared Catastrophe.

Grants: D.-A.D. gratefully acknowledges funding by the DAAD. S.B. is supported by the NFS Grant #DMS-0757581. A.P. is supported by the NSF grant #DMS-0905988.

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1 Introduction and Definition of the Model

We study the mass shell of a non-relativistic spinless quantum particle interacting with the quantized field of relativistic, massless, scalar bosons, where the interaction is linear in the field. This model originated as an effective description of the interaction between non-relativistic nucleons and mesons. It is usually referred to as ‘Nelson model’ since E. Nelson (see [Nel64]) showed how to remove the ultraviolet cut-off that turns off the interaction with the high frequency modes of the field. The limiting Hamiltonian is defined starting from the quadratic form associated with the so-called Gross transformed Hamiltonian. The latter is obtained from the Nelson Hamiltonian through a unitary dressing transformation [Gro62] after subtracting a constant which is divergent in the ultraviolet (UV) limit. This means that only a ground state energy renormalization is necessary in order to define the local interaction. This model for only one nucleon is known as the one particle sector of the translation invariant Nelson model.

In recent years this model has been extensively studied with regard to quantum electrodynamics (QED). In fact, when the bosons are massless particles (i.e. ‘scalar photons’) the model can be seen as a scalar version of the effective theory (non-relativistic QED) that describes a non-relativistic electron interacting with the quantized radiation field. In the study of the translation invariant, massless Nelson model an ultraviolet cut-off of the order of the rest mass energy of the electron is usually imposed. Otherwise relativistic corrections to the electron dynamics and electron-positron pair creation should be taken into account. In spite of these simplifications, the massless Nelson model gives non-perturbative insights on the infrared properties of QED.

It is an interesting mathematical problem to clarify whether the results concerning the infrared region, which have been obtained in presence of an ultraviolet cut-off, can be extended to the ‘renormalized’ Nelson model (i.e. without an ultraviolet cut-off). As presented in [HHS05] these questions do not in general have a straightforward answer.

For the one particle sector of the renormalized Nelson model the study of the mass shell was carried out by Cannon few years after the appearance of Nelson’s paper. In [Can71] it is proven that a perturbed mass shell exists for sufficiently small values of the coupling constant g and in the spectral region (E, P) for $|P| < 1$. Here, E and P are the spectral variables of the Hamiltonian and of the total momentum operator, respectively. In fact, starting from translation invariance, one considers the natural decomposition of the Hilbert space on the spectrum of the total momentum operator and studies the existence of the ground state of the fiber Hamiltonians H_p for $|P| < 1$. In his paper, Cannon relies on the spectral gap of the fiber Hamiltonians induced by a meson mass. The mass shell of the nucleon is then defined by analytic perturbation theory of the ground state eigenvector fiber by fiber for $|P| < 1$ and sufficiently small g . The interaction is in fact a small

perturbation of type B – i.e. in the form sense – with respect to the free Hamiltonian. For this type of perturbation it is in principle possible to control the perturbed spectral projection and to give a meaning to the formal expansion of the ground state vector of the perturbed Hamiltonian. The price for this is a very cumbersome formula (see [Kat95]) making his result almost intractable for applications to scattering theory. As a matter of fact, no explicit expression for the perturbed mass shell is provided in [Can71].

Finally, for the massless Nelson model, the result concerning the existence of the mass shell was extended by Fröhlich to arbitrarily small infrared cut-off with no restriction on the coupling constant. The method used in [Frö73] is based on a lattice approximation of the boson momentum space which is eventually removed, a technique inspired by earlier works of Glimm and Jaffe. However, Fröhlich’s expression for the fiber eigenvectors is only implicit. In recent years the P -dependence of the ground state energy in the massless Nelson model and in non-relativistic QED has been studied in presence of an ultraviolet regularization. [BCFS07] and [Che08] use the isospectral renormalization group whereas [AH10] relies on statistical mechanics methods.

We accomplish three main goals: (1) By using a multiscale technique for small values of the coupling constant and for a fixed infrared cut-off $\kappa > 1$ (in units where the electron mass m , the Planck’s constant \hbar , and the speed of light c all equal one) we first derive the results by Cannon for the massless Nelson model. Rather than using regular perturbation theory for quadratic forms we employ a multiscale technique for operators inspired by [Piz03]. Our construction yields more explicit expressions for the ‘renormalized’ mass shell. In particular, they are amenable to rigorously control the S-matrix elements under the removal of the UV cut-off and to compare them with physicists’ perturbation formulae.

(2) We then show how to construct the mass shell for the renormalized model when the interaction is extended to frequency ranges down to an arbitrarily small infrared cut-off. This result at a small but fixed value of the coupling constant g is beyond the reach of the method employed by Cannon [Can71] because the spectral gap shrinks to zero as the infrared cut-off is removed.

(3) The final part of our analysis concerns the properties of the mass shell in the infrared limit where it is well-known that no *proper* mass shell is present, a fact usually referred to as the *infrared catastrophe*. Following the strategy developed in [Piz03], we implement a suitable Bogolyubov transformation for the field variables corresponding to frequencies below the threshold $\kappa > 1$. In contrast to Gross’ dressing this transformation depends on the P -fiber and is not unitary in the infrared limit. Fiber by fiber, we obtain a transformed Hamiltonian where the interaction is not linear in the field anymore both because of the Gross transformation in the UV region (frequencies larger than κ) and because of the infrared dressing transformation (frequencies smaller than κ). Each transformed Hamiltonian has a ground state in the infrared limit, the construction of which requires a delicate control of the interplay between high and low frequency modes. The control of the mass shell associated with these *unphysical* fiber Hamiltonians is crucial to analyze the infraparticle behavior of the renormalized electron in the massless Nelson model and to provide an asymptotic expansion for the scattering amplitudes in ‘Compton scattering’, free from both ultraviolet and infrared divergences.

Definition of the model. The Hilbert space of the model is

$$\mathcal{H} := L^2(\mathbb{R}^3, \mathbb{C}; dx) \otimes \mathcal{F}(h),$$

where $\mathcal{F}(h)$ is the Fock space of scalar bosons

$$\mathcal{F}(h) := \bigoplus_{j=0}^{\infty} \mathcal{F}^{(j)}, \quad \mathcal{F}^{(0)} := \mathbb{C}, \quad \mathcal{F}^{j \geq 1} := \bigodot_{l=1}^j h, \quad h := L^2(\mathbb{R}^3, \mathbb{C}; dk),$$

where \odot denotes the symmetric tensor product. Let $a(k), a^*(k)$ be the usual Fock space annihilation and creation operators satisfying the canonical commutation relations (CCR)

$$[a(k), a^*(l)] = \delta(k - l), \quad [a(k), a(l)] = [a^*(k), a^*(l)] = 0.$$

The kinematics of the system is described by: (a) The position x and the momentum p of the non-relativistic particle that satisfy the Heisenberg commutation relations. (b) The scalar field Φ and its conjugate momentum where

$$\Phi(y) := \int dk \rho(k) (a(k)e^{iky} + a^*(k)e^{-iky}), \quad \rho(k) := \frac{1}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega(k)}}, \quad \omega(k) := |k|.$$

The dynamics is generated by the Hamiltonian of the Nelson model,

$$H|_{\tau}^{\Lambda} := \frac{p^2}{2} + H^f + g\Phi|_{\tau}^{\Lambda}(x)$$

where

$$H^f := \int dk \omega(k) a^*(k) a(k)$$

is the free field Hamiltonian, and

$$g\Phi|_{\tau}^{\Lambda}(x) := g \int_{\mathcal{B}_{\Lambda} \setminus \mathcal{B}_{\tau}} dk \rho(k) (a(k)e^{ikx} + a^*(k)e^{-ikx}), \quad \rho(k) := \frac{1}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega(k)}}, \quad (1)$$

is the interaction term for $0 \leq \tau < \Lambda < \infty$; here $g \in \mathbb{R}$ is the coupling constant and for the domain of integration we use the notation $\mathcal{B}_{\sigma} := \{k \in \mathbb{R}^3 \mid |k| < \sigma\}$ for any $\sigma > 0$. Note that for $\Lambda = \infty$ the formal expression of the interaction $\Phi|_{\kappa}^{\Lambda}$ is not a well-defined operator on \mathcal{H} because the form factor $\rho(k)$ is not square integrable.

We briefly recall some well-known facts about this model. For $0 \leq \tau < \Lambda < \infty$ the operator $H|_{\tau}^{\Lambda}$ is self-adjoint and its domain coincides with the one of $H_0 := \frac{p^2}{2} + H^f$ (see also Proposition 1.1 below). The total momentum operator of the system is

$$P := p + P^f := p + \int dk k a^*(k) a(k)$$

where P^f is the field momentum. Due to translational invariance of the system the Hamiltonian and the total momentum operator commute. Hence, the Hilbert space \mathcal{H} can be decomposed on the joint spectrum of the three components of the total momentum operator, i.e.

$$\mathcal{H} = \int^{\oplus} dP \mathcal{H}_P$$

where \mathcal{H}_P is a copy of the Fock space \mathcal{F} carrying the (Fock) representation corresponding to annihilation and creation operators

$$b(k) := a(k)e^{ikx}, \quad b^*(k) := a^*(k)e^{-ikx}.$$

We will use the same symbol \mathcal{F} for all Fock spaces. The fiber Hamiltonian can be expressed as

$$H_P|_\tau^\Lambda := \frac{1}{2}(P - P^f)^2 + H^f + g \int_{\mathcal{B}_\Lambda \setminus \mathcal{B}_\tau} dk \rho(k) (b(k) + b^*(k)).$$

By construction, the fiber Hamiltonian maps its domain in \mathcal{H}_P into \mathcal{H}_P . Finally, for later use we define

$$H_{P,0} := \frac{(P - P^f)^2}{2} + H^f, \quad \Delta H_P|_\tau^\Lambda := H_P|_\tau^\Lambda - H_{P,0}. \quad (2)$$

The Gross transformation. We use a frequency

$$1 < \kappa < 2$$

to separate the ultraviolet and the infrared regimes. The renormalization of the Hamiltonian must cure the divergence which appears in the second order correction to the ground state energy as $\Lambda \rightarrow \infty$. This logarithmically divergent term

$$V_{\text{self}}|_\kappa^\Lambda := -\frac{g^2}{[2(2\pi)^3]} \int_{\mathcal{B}_\Lambda \setminus \mathcal{B}_\kappa} dk \frac{1}{|k| \left[\frac{|k|^2}{2} + |k| \right]} \quad (3)$$

can be separated from the rest of the Hamiltonian by a Bogolyubov transformation $e^{-T|_\kappa^\Lambda}$, acting on all frequencies above κ , whose skew-adjoint generator is given by

$$T|_\kappa^\Lambda := \int_{\mathcal{B}_\Lambda \setminus \mathcal{B}_\kappa} dk \beta(k) (b(k) - b^*(k)), \quad \beta(k) := -g \frac{\rho(k)}{\frac{|k|^2}{2} + \omega(k)}. \quad (4)$$

Note that for any $1 < \kappa < \Lambda \leq \infty$, the operators $T|_\kappa^\Lambda$, $T^*|_\kappa^\Lambda$ are well-defined on $D(H_{P,0})$. For $1 < \kappa < \Lambda < \infty$ the Hamiltonian $H_P|_\kappa^\Lambda$ transforms as follows:

$$H'_P|_\kappa^\Lambda := e^{T|_\kappa^\Lambda} H_P|_\kappa^\Lambda e^{-T|_\kappa^\Lambda} - V_{\text{self}}|_\kappa^\Lambda \quad (5)$$

$$= \frac{1}{2}(P - P^f)^2 + H^f + \frac{1}{2}[(B|_\kappa^\Lambda)^2 + (B^*|_\kappa^\Lambda)^2] + B^*|_\kappa^\Lambda \cdot B|_\kappa^\Lambda - (P - P^f) \cdot B|_\kappa^\Lambda - B^*|_\kappa^\Lambda \cdot (P - P^f) \quad (6)$$

where

$$B|_\kappa^\Lambda := \int_{\mathcal{B}_\Lambda \setminus \mathcal{B}_\kappa} dk k \beta(k) b(k). \quad (7)$$

It is important to note that the operator equality (6) holds on $D(H_{P,0})$ as proven in [Nel64, Lemma 3]. In the following sections we will study the renormalized Hamiltonian

$$H'_P|_\kappa^\Lambda + g\Phi|_\tau^\kappa \quad (8)$$

The proofs of [Nel64, Lemma 2 and 3] imply:

Proposition 1.1. *For $0 \leq \tau < \Lambda < \infty$, the operators $H_P|_\tau^\Lambda$ and $H_P|_\kappa^\Lambda + g\Phi|_\tau^\kappa$ are self-adjoint and their domain coincide with the one of $H_{P,0}$.*

By [Nel64, Main Theorem] there exists an ultraviolet renormalized Hamiltonian:

Theorem 1.2. *For all $\tau \geq 0$, there is a unique self-adjoint operator $H_P|_\tau^\infty$ on \mathcal{F} that generates the unitary group defined by*

$$e^{-itH_P|_\tau^\infty} := \text{s-lim}_{\Lambda \rightarrow \infty} e^{-it(H_P|_\tau^\Lambda - V_{\text{self}}|_\kappa^\Lambda)}, \quad t \in \mathbb{R}.$$

The domain of $H_P|_\tau^\infty$ is a dense subset of the domain of $H_{P,0}^{1/2}$, and $H_P|_\tau^\infty$ is bounded from below.

However, we will not make use of Theorem 1.2. In the case of $|P| < P_{\max}$ defined in (9) and for sufficiently small $|g|$ this result will follow from our multiscale analysis.

2 Main Results

Since the particle is non-relativistic we restrict the total momentum to the ball

$$|P| \leq P_{\max} := \frac{1}{4}. \quad (9)$$

The ultraviolet and infrared scaling. We shall introduce a scaling that divides the interaction term into slices of boson momenta for which, step by step, we apply analytic perturbation theory. In the ultraviolet regime, this scaling is defined by the sequence

$$\sigma_n := \kappa\beta^n, \quad 1 < \beta, \quad n \in \mathbb{N},$$

while in the infrared regime we use

$$\tau_m := \kappa\gamma^m, \quad 0 < \gamma < \frac{1}{2}, \quad m \in \mathbb{N}.$$

With respect to these scalings we shall use the following notation for Hamiltonians and Fock spaces:

IR	UV	Hamiltonian	Fock space
κ	σ_n	$H_P _0^n := H_P _\kappa^{\sigma_n}$	$\mathcal{F} _0^n := \mathcal{F}(L^2(\mathcal{B}_{\sigma_n} \setminus \mathcal{B}_\kappa))$
τ_m	σ_n	$H_P _m^n := H_P _0^n + g\Phi _{\tau_m}^\kappa$	$\mathcal{F} _m^n := \mathcal{F}(L^2(\mathcal{B}_{\sigma_n} \setminus \mathcal{B}_{\tau_m}))$

The normalized vacuum vector in each of these Fock spaces is denoted by the same symbol Ω . We shall exclusively use the index n to denote the ultraviolet cut-off σ_n and the index m to denote the infrared cut-off τ_m , e.g.

$$\mathcal{F}|_{n-1}^n := \mathcal{F}(L^2(\mathcal{B}_{\sigma_n} \setminus \mathcal{B}_{\sigma_{n-1}})), \quad \mathcal{F}|_m^{m-1} := \mathcal{F}(L^2(\mathcal{B}_{\tau_{m-1}} \setminus \mathcal{B}_{\tau_m})).$$

For a vector ψ in $\mathcal{F}|_0^{n-1}$ and an operator O on $\mathcal{F}|_0^{n-1}$ we shall use the same symbol to denote the vector $\psi \otimes \Omega$ in $\mathcal{F}|_0^n$ and the operator $O \otimes \mathbb{1}_{\mathcal{F}|_{n-1}^n}$ on $\mathcal{F}|_0^n$, respectively.

Moreover, the Fock space slices and the related interaction terms are given by

	Slice	Interaction	Fock space
UV	$[\sigma_{n-1}, \sigma_n)$	$\Delta H'_P _{n-1}^n := H'_P _0^n - H'_P _0^{n-1}$	$\mathcal{F} _{n-1}^n$
IR	$(\tau_m, \tau_{m-1}]$	$g\Phi _m^{m-1} := g\Phi _{\tau_m}^{\tau_{m-1}}$	$\mathcal{F} _m^{m-1}$

Similarly we shall use $|_m^n, |_{n-1}^n, |_m^{m-1}$ instead of $|\sigma_n, |\sigma_{n-1}, |\tau_{m-1}$, respectively, as short-hand notation to denote the range of boson momenta associated with the interaction.

For a self-adjoint operator A which is bounded from below we define the spectral gap as

$$\text{Gap}(A) := \inf\{\text{Spec}(A) \setminus \{\inf \text{Spec}(A)\}\} - \inf \text{Spec}(A).$$

Moreover, we denote

$$E_P|_m^n := \inf \text{Spec}(H_P|_m^n \upharpoonright \mathcal{F}|_m^n), \quad E'_P|_m^n := \inf \text{Spec}(H'_P|_m^n \upharpoonright \mathcal{F}|_m^n) = E_P|_m^n - V_{\text{self}}|_0^n \quad (10)$$

where $\text{Spec}(A \upharpoonright X)$ denotes the spectrum of the linear operator A restricted to the subspace X . If $E'_P|_m^n$ is a non-degenerate eigenvalue of the Hamiltonian $H'_P|_m^n$ we shall denote a (possibly unnormalized) corresponding eigenvector by $\Psi'_P|_m^n$. In this situation we have

$$\text{Gap}(H'_P|_m^n \upharpoonright \mathcal{F}|_m^n) = \inf_{\psi \perp \Psi'_P|_m^n} \langle H'_P|_m^n - E'_P|_m^n \rangle_\psi$$

where the infimum is taken over the vectors ψ in the domain of $H'_P|_m^n \upharpoonright \mathcal{F}|_m^n$, and we have used the notation

$$\langle A \rangle_\psi = \frac{\langle \psi, A\psi \rangle}{\langle \psi, \psi \rangle}$$

for any operator A and $\psi \in D(A)$.

The Mass Shell of $H'_P|_0^\infty$. The multiscale perturbation theory that we use here relies on the control of the spectral gap as more and more slices of the interaction Hamiltonian are added. In the construction of the mass shell eigenvectors one observes a major difference between removing the ultraviolet and the infrared cut-off. In the infrared limit the main problem is that the gap closes and the infimum of the spectrum is not an eigenvalue anymore (see [Piz03]). In the ultraviolet limit the main problem is that the whole spectrum moves towards $-\infty$. The latter is caused by the well-known logarithmic divergence in (3). In order to gain control on the gap it is necessary to extract this divergent term which, as it is also well-known, can be accomplished via the Gross transformation. At first, we shall therefore apply the multiscale perturbation theory to the Gross transformed Hamiltonians $H'_P|_0^n$, and then use unitarity to inherit all results for the back-transformed Nelson Hamiltonians

$$H_P|_0^n := e^{-T|_0^n} H'_P|_0^n e^{T|_0^n} + V_{\text{self}}|_0^n, \quad n \in \mathbb{N}.$$

The iterative analytic perturbation theory, which was successfully applied for the infrared regime [Piz03], can be adapted to the ultraviolet regime using the following induction:

Suppose that, for a given and appropriately chosen real sequence $(\xi_n)_{n \in \mathbb{N}}$ bounded from below by a positive constant, we know that the following holds for the $(n-1)$ -th step of the induction:

(i) $\Psi'_{p|_0^{n-1}}$ is the unique ground state of $H'_{p|_0^{n-1}}$ with energy $E'_{p|_0^{n-1}}$.

(ii) $\text{Gap}(H'_{p|_0^{n-1}} \upharpoonright \mathcal{F}|_0^{n-1}) \geq \xi_{n-1}$.

In order to show the induction step $(n-1) \Rightarrow n$, we first estimate the new spectral gap while adding the slice $\mathcal{F}|_{n-1}^n$ of boson Fock space without modifying the Hamiltonian. An a priori variational argument yields $\text{Gap}(H'_{p|_0^{n-1}} \upharpoonright \mathcal{F}|_0^n) \geq \xi_{n-1}$. With this at hand we apply analytic perturbation theory à la Kato to construct the ground state of $H'_{p|_0^n} \upharpoonright \mathcal{F}|_0^n$. More precisely, we show that the Neumann series of the resolvent

$$\frac{1}{H'_{p|_0^n} - z} = \frac{1}{H'_{p|_0^{n-1}} - z} \sum_{j=0}^{\infty} [-\Delta H'_{p|_0^{n-1}} \frac{1}{H'_{p|_0^{n-1}} - z}]^j \quad (11)$$

is well-defined for all z in the domain

$$\frac{1}{2}\xi_n \leq |E'_{p|_0^{n-1}} - z| \leq \xi_n < \xi_{n-1}.$$

Step by step we show the convergence of the Neumann series for a sufficiently small $|g|$ (and β sufficiently close to one) but uniformly in n . In the control of the resolvent in (11) a convenient definition of $(\xi_n)_{n \in \mathbb{N}}$ turns out to be crucial. Kato's perturbation theory ensures the existence of a projection $Q'_{p|_0^n}$ onto the unique ground state $\Psi'_{p|_0^n}$ with eigenvalue $E'_{p|_0^n}$. Since an a priori variational argument yields $E'_{p|_0^n} \leq E'_{p|_0^{n-1}}$, we conclude that $\text{Gap}(H'_{p|_0^n} \upharpoonright \mathcal{F}|_0^n) \geq \xi_n$.

This way we construct a convergent sequence of ground states corresponding to $H'_{p|_0^n}$, $n \in \mathbb{N}$,

$$\Psi'_{p|_0^n} := Q'_{p|_0^n} Q'_{p|_0^{n-1}} \cdots Q'_{p|_0^1} \Omega$$

where Ω is the ground state of $H'_{p,0}$. The projections $Q'_{p|_0^n}$ will be given explicitly in (76). Finally, the unitarity of the Gross transformation implies that

$$\Psi_{p|_0^n} := e^{-T|_0^n} \Psi'_{p|_0^n}, \quad n \in \mathbb{N},$$

is a sequence of ground states of $H_{p|_0^n}$ that also converges, say to a $\Psi_{p|_0^\infty} \in \mathcal{F}$. Furthermore, we prove the convergence of $H'_{p|_0^n}$ in the norm resolvent sense to a limiting Hamiltonian $H_{p|_0^\infty}$, the unique ground state of which is $\Psi_{p|_0^\infty}$. Precisely, we prove:

Theorem 2.1. *Let $|P| \leq P_{\max}$. There is a constant $g_{\max} > 0$ such that for all $|g| < g_{\max}$ the following holds true:*

- (i) *The sequence of operators $(H_{p|_0^n} - V_{\text{self}|_0^n})_{n \in \mathbb{N}}$ converges in the norm resolvent sense to a self-adjoint operator $H_{p|_0^\infty}$ acting on \mathcal{F} .*
- (ii) *The limit $\Psi_{p|_0^\infty} := \lim_{n \rightarrow \infty} \Psi_{p|_0^n}$ exists in \mathcal{F} and is non-zero.*
- (iii) *$E_{p|_0^\infty} := \lim_{n \rightarrow \infty} (E_{p|_0^n} - V_{\text{self}|_0^n})$ exists.*
- (iv) *$E_{p|_0^\infty}$ is the non-degenerate ground state energy of the Hamiltonian $H_{p|_0^\infty}$ with corresponding ground state $\Psi_{p|_0^\infty}$. Moreover, the spectral gap of $H_{p|_0^\infty} \upharpoonright \mathcal{F}|_0^\infty$ is bounded from below by $\frac{1}{16}\kappa$.*

The Mass Shell of $H_p|_m^n$ for $m \in \mathbb{N}$. Starting from the ground states $\Psi_p|_0^n$ of the Hamiltonian $H_p|_0^n$, we continue to add interaction slices $g\Phi|_{\tau_m}^{m-1}$, $m \in \mathbb{N}$, now below the frequency κ and construct the family of ground states $\Psi_p|_m^n$ of the Hamiltonians $H_p|_m^n$ with eigenvalue $E_p|_m^n$, i.e.

$$H_p|_m^n \Psi_p|_m^n = E_p|_m^n \Psi_p|_m^n.$$

For arbitrarily large but fixed $m \in \mathbb{N}$, we prove results analogous to Theorem 2.1: Norm resolvent convergence of $(H_p|_m^n)_{n \in \mathbb{N}}$ is shown in Lemma 5.2. The existence of $\Psi_p|_m^\infty$, $m \in \mathbb{N}$, is shown in Theorem 5.8. In particular, the spectral gap of $H_p|_m^n$ is bounded from below by a constant times τ_m uniformly for all $n \in \mathbb{N} \cup \{\infty\}$. This is proven in Lemma 5.5.

The Mass Shell of $H_p^{W'}|_\infty$. As it is well-known (see [Frö73, Piz03]), for every $n \in \mathbb{N} \cup \{\infty\}$ the ground state $\frac{\Psi_p|_m^n}{\|\Psi_p|_m^n\|}$ weakly converge to zero as $m \rightarrow \infty$. This is linked to the infamous infrared catastrophe problem in QED. In fact, in the infrared limit the interaction turns out to be *marginal* according to renormalization group terminology. On the other hand it was proven in [Frö73] that

$$b(k)\Psi_p|_m^n = g \rho(k) \frac{1}{E_p|_m^n - |k| - H_{p-k}|_m^n} \Psi_p|_m^n \quad (12)$$

which implies that

$$b(k)\Psi_p|_m^n \approx \alpha_m(\nabla E_p|_m^n, k) \Psi_p|_m^n, \quad \alpha_m(Q, k) := -g \frac{\rho(k)}{\omega(k)} \frac{\mathbb{1}_{\mathcal{B}_\kappa \setminus \mathcal{B}_{\tau_m}}(k)}{1 - \widehat{k} \cdot Q} \quad (13)$$

up to higher order terms as $k \rightarrow 0$. This motivates a strategy to analyze the infrared limit by using the Bogolyubov transformation $W_m(\nabla E_p|_m^n)$ defined as follows: for $Q \in \mathbb{R}^3$, $|Q| < 1$,

$$W_m(Q) b^\#(k) W_m(Q)^* := b^\#(k) + \alpha_m(Q, k) \quad b^\#(k) = b(k), b^*(k). \quad (14)$$

Instead of studying $H_p|_m^n$ directly one considers the transformed Hamiltonian

$$H_p^{W'}|_m^n := W_m(\nabla E_p|_m^n) H_p|_m^n W_m(\nabla E_p|_m^n)^*. \quad (15)$$

Note that the transformation acts non-trivially only on boson momenta below κ . For any finite m , the operator $W_m(Q)$ is unitary but this property does not hold anymore in the limit $m \rightarrow \infty$. Furthermore, for $Q \neq Q'$ the function $\alpha_m(Q, k) - \alpha_m(Q', k)$ is not square integrable as $m \rightarrow \infty$.

Most importantly, the interaction term

$$H_p^{W'}|_m^n - H_{p,0} \quad (16)$$

of the transformed Hamiltonian is now *superficially marginal* in the infrared limit, in contrast to the interaction $H_p|_m^n - H_{p,0}$. At a fixed ultraviolet cut-off and at a small coupling constant g , it has been proven in [Piz03] that the sequence of ground states $(\phi_p|_m^n)_{m \in \mathbb{N}}$, i.e.

$$H_p^{W'}|_m^n \phi_p|_m^n = E_p|_m^n \phi_p|_m^n, \quad (17)$$

converges in the limit $m \rightarrow \infty$ while the spectral gap closes. Consequently, infrared asymptotic freedom holds. This result requires a sophisticated proof by induction. In the present paper we

prove the same result while simultaneously removing the ultraviolet cut-off. Before sketching the main technical difficulties in dealing with the construction of the states $\phi_P|_\infty$ let us briefly explain their physical relevance.

With the states $\phi_P|_m^n$ and the Bogolyubov transformation $W_m(\nabla E'_P|_m^n)$ at hand it is possible to control the properties of the physical mass shell given by the states $\Psi'_P|_m^n$ in the infrared limit, i.e. $m \rightarrow \infty$, namely the dependence on the total momentum P . This spectral information represents the key ingredient to construct the scattering states for the so-called *infraparticles* (see [Piz03] and [CFP09]). The QED analogue of the transformation of the field variables in (14) is related to the Liénard-Wiechert fields carried by the charged particle and to the infrared radiation emitted in Compton scattering; see [CFP09] for precise mathematical statements.

More technically, while simultaneously removing the infrared and the ultraviolet cut-off in $\phi_P|_m^n$ a major difficulty arises in the induction mentioned above. In fact, the proof of the limit of $\phi_P|_m^n$ as $m \rightarrow \infty$ relies on a suitable rearrangement of the terms in the Hamiltonian $H_P^{W'}|_m^n$ given by

$$H_P^{W'}|_m^n = \frac{1}{2}\Gamma_P|_m^n{}^2 + H^f - \nabla E'_P|_m^n \cdot P^f + C_{P,m}^{(n)} + R_P|_m^n, \quad (18)$$

see (85) in Section 6, where the vector operator $\Gamma_P|_m^n$ has the crucial property

$$\langle \phi_P|_m^n, \Gamma_P|_m^n \phi_P|_m^n \rangle = 0. \quad (19)$$

However, the operator $\Gamma_P|_m^n$ is ill-defined in the limit $n \rightarrow \infty$. This suggests the following strategy for the simultaneous removal of the cut-offs, for sufficiently small g but uniform in n and m :

- (i) First show that $(\phi_P|_m^n)_{m \in \mathbb{N}}$ is a Cauchy sequence uniformly in n ;
- (ii) then provide bounds of the form

$$\|\phi_P|_m^n - \phi_P|_m^{n-1}\| \leq f_1(n, m), \quad (20)$$

and

$$|\nabla E'_P|_m^n - \nabla E'_P|_m^{n-1}| \leq f_2(n, m), \quad (21)$$

where $f_1(n, m)$ and $f_2(n, m)$ are such that for the scaling $n(m) := \alpha m$ with α sufficiently large both $(\phi_P|_m^{n(m)})_{n \in \mathbb{N}}$ and $(\nabla E'_P|_m^{n(m)})_{n \in \mathbb{N}}$ are Cauchy sequences.

This program will be carried out in Sections 6 and 7. It will yield the second main result:

Theorem 2.2. *Let $|P| \leq P_{\max}$. For $|g|$ sufficiently small the following holds true:*

- (i) *There exists an $\alpha_{\min} > 0$ such that for any integer $\alpha' > \alpha_{\min}$ and $n(m) = \alpha' m$, the limit*

$$\phi_P|_\infty := \lim_{m \rightarrow \infty} \phi_P|_m^{n(m)}$$

exists in \mathcal{F} and is non-zero.

- (ii) *$E'_{P,\infty} := \lim_{m \rightarrow \infty} E'_P|_m^\infty$ exists and is the ground state energy corresponding to the eigenvector $\phi_P|_\infty$ of the self-adjoint operator*

$$H_P^{W'}|_\infty := \lim_{m \rightarrow \infty} H_P^{W'}|_m^{n(m)},$$

where the limit is understood in the norm resolvent sense.

At this point, we emphasize that at least within the scope of the presented multiscale technique, the given scaling to remove both UV and IR cut-offs simultaneously is natural. The method indeed relies on the control of the spectral gap, and as the gap closes in the IR limit, the UV limit must be taken at a comparatively fast enough rate.

For the notation throughout this paper, the reader is advised to consult the list below.

Notation.

1. By convention $0 \notin \mathbb{N}$.
2. The symbol C denotes any universal constant. Any appearing C is independent of the indices m and n and of all parameters in the paper, i.e. g, β, γ and ζ , at least in prescribed neighborhoods.
3. The bars $|\cdot|$, $\|\cdot\|$ denote the euclidean and the Fock space norm, respectively. The brackets $\langle \cdot, \cdot \rangle$ denote the scalar product of vectors in \mathcal{F} . Given a subspace $\mathcal{K} \subseteq \mathcal{F}$ and an operator A on \mathcal{F} we use the notation

$$\|A\|_{\mathcal{K}} = \|A \upharpoonright \mathcal{K}\|.$$

4. For a vector operator $A = (A^{(1)}, A^{(2)}, A^{(3)})$ with components $A^{(i)} : D(A^{(i)}) \rightarrow \mathcal{F}$, $1 \leq i \leq 3$, we use the notation

$$\|A\psi\|^2 = \sum_{i=1}^3 \|A^{(i)}\psi\|^2.$$

3 Tools

We recall some standard operator inequalities which are frequently used. For every square integrable function f the estimates

$$\begin{aligned} \left\| \int_{\mathcal{B}_\Lambda \setminus \mathcal{B}_\tau} dk f(k) b(k) \psi \right\| &\leq \left(\int_{\mathcal{B}_\Lambda \setminus \mathcal{B}_\tau} dk \left| \frac{f(k)}{\sqrt{|k|}} \right|^2 \right)^{1/2} \| (H^f|_\tau^\Lambda)^{1/2} \psi \|, \\ \left\| \int_{\mathcal{B}_\Lambda \setminus \mathcal{B}_\tau} dk f(k) b^*(k) \psi \right\| &\leq \left(\int_{\mathcal{B}_\Lambda \setminus \mathcal{B}_\tau} dk \left| \frac{f(k)}{\sqrt{|k|}} \right|^2 \right)^{1/2} \| (H^f|_\tau^\Lambda)^{1/2} \psi \| \\ &\quad + \left(\int_{\mathcal{B}_\Lambda \setminus \mathcal{B}_\tau} dk |f(k)|^2 \right)^{1/2} \|\psi\| \end{aligned} \tag{22}$$

hold true for all $0 \leq \tau < \Lambda \leq \infty$ and ψ in the domain of $H_{P,0}^{1/2}$ whenever the integrals on the right-hand side of (22) are well defined.

The following two results are crucial ingredients in the proofs presented in the next sections. The first one, Theorem 3.1, is a classical result by L. Gross that turns out to be the main non-perturbative ingredient for the gap estimates that we obtain by iterative analytic perturbation theory; see Sections 4 and 5.

Theorem 3.1. For $0 \leq \tau < \Lambda < \infty$ and all $P \in \mathbb{R}^3$ the ground state energies $E_P|_\tau^\Lambda := \inf \text{Spec} \left(H_P|_\tau^\Lambda \right)$ fulfill $E_0|_\tau^\Lambda \leq E_P|_\tau^\Lambda$.

Proof. See [Gro72, Theorem 8]. \square

The second one, Lemma 3.2, plays a role in Sections 5, 6, 7 where we consider the interaction both in the ultraviolet and in the infrared regime. It is a crucial ingredient to prove statements (i), (ii) in Corollary 5.4. We stress that the multiscale technique which we apply in Section 4 to remove the ultraviolet cut-off at $m = 0$ does not refer to Corollary 5.4 (i),(ii), and only relies on Theorem 3.1 and on a weaker estimate given in (48) that follows from (22).

Lemma 3.2. There exist finite constants $c_a, c_b > 0$ such that

$$\langle \psi, H_{P,0} \psi \rangle \leq \frac{1}{1 - |g|c_a} \left[\langle \psi, H'_{P,m}|_m^n \psi \rangle + |g|c_b \langle \psi, \psi \rangle \right] \quad (23)$$

for $|g| \leq 1, \frac{1}{c_a}$ and $\psi \in D(H_{P,0}^{1/2})$ with $m, n \in \mathbb{N}$.

Proof. See Appendix A. \square

4 Ground States of the Gross Transformed Hamiltonians $H'_P|_0^\infty$

This section provides the proof of Theorem 2.1 in Section 2. We start by introducing a sequence of gap bounds.

Definition 4.1. We define the sequence of gap bounds

$$\xi_n := \frac{1}{8}\kappa \left(1 - \sum_{j=1}^n \Delta \xi_j \right), \quad \Delta \xi_n := \frac{(\beta - 1)^2}{2\beta} \frac{n}{\beta^n} \quad (24)$$

for $n \in \mathbb{N}$ with the scaling parameter $\beta > 1$. Furthermore, we impose the constraint

$$|g| \leq (\beta - 1), \quad 1 < \beta < 2. \quad (25)$$

The definition of the sequence of gap bounds $(\xi_n)_{n \in \mathbb{N}}$ in (24) will be motivated in Lemma 4.5. Note that $\sum_{j=1}^\infty \Delta \xi_j = \frac{1}{2}$ implies

$$\frac{1}{16}\kappa \leq \xi_n \leq \frac{1}{8}\kappa. \quad (26)$$

Remark 4.2. In this section the constraints $|P| < P_{\max}$ and $1 < \kappa < 2$ are implicitly assumed.

Lemma 4.3. For an integer $n > 1$ assume:

- (i) $E'_P|_0^{n-1}$ is the non-degenerate eigenvalue of $H'_P|_0^{n-1} \upharpoonright \mathcal{F}|_0^{n-1}$ with eigenvector $\Psi'_P|_0^{n-1}$.
- (ii) $\text{Gap} \left(H'_P|_0^{n-1} \upharpoonright \mathcal{F}|_0^{n-1} \right) \geq \xi_{n-1}$.
- (iii) $E'_P|_0^{n-1}$ is differentiable in P and $|\nabla E'_P|_0^{n-1}| \leq C_{\nabla E} \equiv \frac{3}{4}$.

This implies that $E'_{p|_0}|^{n-1}$ is also the non-degenerate ground state energy of $H'_{p|_0}|^{n-1} \upharpoonright \mathcal{F}|_0^n$ with eigenvector $\Psi'_{p|_0}|^{n-1} \otimes \Omega$. Furthermore,

$$\text{Gap}\left(H'_{p|_0}|^{n-1} \upharpoonright \mathcal{F}|_0^n\right) \geq \inf_{\mathcal{F}|_0^n \ni \psi \perp \Psi'_{p|_0}|^{n-1} \otimes \Omega} \left\langle H'_{p|_0}|^{n-1} - \theta H^f|_{n-1}^n - E'_{p|_0}|^{n-1} \right\rangle_\psi \geq \xi_{n-1} \quad (27)$$

where $0 < \theta < \frac{1}{8}$ and the infimum is taken over $\psi \in D(H_{p,0})$.

Proof. Using (i), a direct computation yields

$$H'_{p|_0}|^{n-1}(\Psi'_{p|_0}|^{n-1} \otimes \Omega) = E'_{p|_0}|^{n-1}(\Psi'_{p|_0}|^{n-1} \otimes \Omega)$$

as the interaction is cut off at σ_{n-1} . Hence, $E'_{p|_0}|^{n-1}$ is an eigenvalue of $H'_{p|_0}|^{n-1} \upharpoonright \mathcal{F}|_0^n$ with eigenvector $\Psi'_{p|_0}|^{n-1} \otimes \Omega$. Let us consider

$$\inf_{\mathcal{F}|_0^n \ni \psi \perp \Psi'_{p|_0}|^{n-1} \otimes \Omega} \left\langle H'_{p|_0}|^{n-1} - E'_{p|_0}|^{n-1} \right\rangle_\psi. \quad (28)$$

As the Gross transformation is unitary and does not affect $\mathcal{F}|_{n-1}^n$, and since $H^f|_{n-1}^n$ is positive, we have

$$(28) \geq \inf_{\mathcal{F}|_0^n \ni \psi \perp \Psi'_{p|_0}|^{n-1} \otimes \Omega} \left\langle H'_{p|_0}|^{n-1} - \theta H^f|_{n-1}^n - E'_{p|_0}|^{n-1} \right\rangle_\psi. \quad (29)$$

We subtract the term $\theta H^f|_{n-1}^n$ for a technical reason which will become clear in Lemma 4.5.

Now, the right-hand side of (29) is bounded from below by

$$\min \left\{ \text{Gap}\left(H'_{p|_0}|^{n-1} \upharpoonright \mathcal{F}|_0^n\right), \inf_{\psi=\varphi \otimes \eta} \left\langle H'_{p|_0}|^{n-1} - \theta H^f|_{n-1}^n - E'_{p|_0}|^{n-1} \right\rangle_\psi \right\},$$

where $\varphi \in \mathcal{F}|_0^{n-1}$, $\eta \in \mathcal{F}|_{n-1}^n$, $\varphi \otimes \eta$ belongs to $D(H_{p,0})$ and η is a vector with a definite, strictly positive number of bosons. For $m \geq 1$ bosons in the vector η we estimate

$$\begin{aligned} & \inf_{\psi=\varphi \otimes \eta} \left\langle H'_{p|_0}|^{n-1} - \theta H^f|_{n-1}^n - E'_{p|_0}|^{n-1} \right\rangle_\psi \\ & \geq \inf_{\varphi, k_j \in [\sigma_{n-1}, \sigma_n]} \left\langle \frac{1}{2} \left(P - P^f - \sum_{j=1}^m k_j \right)^2 + H^f + g \Phi|_0^{n-1} + (1 - \theta) \sum_{j=1}^m |k_j| - E'_{p|_0}|^{n-1} \right\rangle_\varphi \\ & \geq \inf_{k_j \in [\sigma_{n-1}, \sigma_n]} \left[(1 - \theta) \sum_{j=1}^m |k_j| + E_{P - \sum_{j=1}^m k_j}|_0^{n-1} - E'_{p|_0}|^{n-1} \right] \end{aligned} \quad (30)$$

$$\geq (1 - \theta - C_{\nabla E}) \sigma_{n-1} \geq \frac{1}{8} \kappa \quad (31)$$

where the steps (30) and (31) follow from:

1. $\sigma_{n-1} \geq \kappa$, $0 < \theta < \frac{1}{8}$ and $C_{\nabla E} = \frac{3}{4}$.

2. The estimate

$$E_{P - \sum_{j=1}^m k_j}|_0^{n-1} - E'_{p|_0}|^{n-1} = E_{P - \sum_{j=1}^m k_j}|_0^{n-1} - E_0|_0^{n-1} + E_0|_0^{n-1} - E'_{p|_0}|^{n-1} \geq E_0|_0^{n-1} - E'_{p|_0}|^{n-1}$$

which holds by Theorem 3.1.

3. The estimate

$$E'_{0|0}{}^{n-1} - E_{P|0}{}^{n-1} \geq - \sup_{|Q| \leq P_{\max}} |\nabla E'_{Q|0}| \geq -C_{\nabla E}$$

since $E'_{P|0}{}^{n-1}$ is differentiable in P and $|P| < 1$.

First, this implies that (28) is bounded from below by $\min \left\{ \xi_{n-1}, \frac{\kappa}{8} \right\} = \xi_{n-1}$; see (26). Second, it turns out that $\Psi'_{P|0}{}^{n-1}$ is the non-degenerate ground state of $H'_{P|0}{}^{n-1} \upharpoonright \mathcal{F}|_0^n$ with

$$\text{Gap} \left(H'_{P|0}{}^{n-1} \upharpoonright \mathcal{F}|_0^n \right) \geq \xi_{n-1}.$$

□

Remark 4.4. Under the assumptions of Lemma 4.3 it follows that for $j, n \in \mathbb{N}$

$$E'_{P|0}{}^n = \inf \text{Spec} \left(H'_{P|0}{}^n \upharpoonright \mathcal{F}|_0^n \right) = \inf \text{Spec} \left(H'_{P|0}{}^n \upharpoonright \mathcal{F}_{n+j}|_0^n \right).$$

Lemma 4.5. Let $n \geq 1$. For $n = 1$, set $H'_{P|0}{}^{n-1} := H'_{P,0}$, $E'_{P|0}{}^{n-1} := P^2/2$, and $\xi_{n-1} := \kappa/2$. Assume that for some universal constant $C_{E'}$ the bound $|E'_{P|0}{}^{n-1}| < C_{E'}$ holds true. Then there exist $\beta_{\max} > 1$ and $g_{\max} > 0$ such that, for all $1 < \beta \leq \beta_{\max}$ and $|g| \leq g_{\max}$, the assumptions (i), (ii) in Lemma 4.3 imply that

$$\frac{1}{H'_{P|0}{}^n - z} \upharpoonright \mathcal{F}|_0^n, \quad \frac{\xi_n}{2} \leq |E'_{P|0}{}^{n-1} - z| \leq \xi_n, \quad (32)$$

is well-defined.

Proof. Let z be in the domain given in (32). In order to control the expansion of the resolvent $(H'_{P|0}{}^n - z)^{-1}$, i.e.

$$\frac{1}{H'_{P|0}{}^{n-1} - z} \sum_{j=0}^{\infty} \left[-\Delta H'_{P|0}{}^{n-1} \frac{1}{H'_{P|0}{}^{n-1} - z} \right]^j \upharpoonright \mathcal{F}|_0^n,$$

it is sufficient to prove that

$$\left\| \left(\frac{1}{H'_{P|0}{}^{n-1} - z} \right)^{1/2} \Delta H'_{P|0}{}^{n-1} \left(\frac{1}{H'_{P|0}{}^{n-1} - z} \right)^{1/2} \right\|_{\mathcal{F}|_0^n} < 1. \quad (33)$$

As we shall show now, this can be achieved by a convenient choice of β and g (uniformly in n) using the gap bounds $(\xi_n)_{n \in \mathbb{N}}$ from Definition 4.1. We can express the interaction term by

$$\begin{aligned} \Delta H'_{P|0}{}^n &= \frac{1}{2} \left((B|_{n-1}^n)^2 + (B^*|_{n-1}^n)^2 \right) + B|_0^{n-1} \cdot B|_{n-1}^n + B^*|_{n-1}^n \cdot B^*|_0^{n-1} \\ &\quad - (P - P^f) \cdot B|_{n-1}^n - B^*|_{n-1}^n \cdot (P - P^f) \\ &\quad + B^*|_{n-1}^n \cdot B|_{n-1}^n + B^*|_0^{n-1} \cdot B|_{n-1}^n + B^*|_{n-1}^n \cdot B|_0^{n-1}. \end{aligned} \quad (34)$$

Hence, the left-hand side of (33) is bounded by

$$\left\| B_{n-1}^n \left(\frac{1}{H_{P|0}^{'n-1} - z} \right)^{1/2} \right\|_{\mathcal{F}_{|0}^n} \times \quad (35)$$

$$\times \left[\left\| B_{n-1}^{*n} \left(\frac{1}{H_{P|0}^{'n-1} - z} \right)^{1/2} \right\|_{\mathcal{F}_{|0}^n} + \left\| B_{n-1}^n \left(\frac{1}{H_{P|0}^{'n-1} - z} \right)^{1/2} \right\|_{\mathcal{F}_{|0}^n} \right] \quad (36)$$

$$+ 2 \left\| B_{|0}^{*n-1} \left(\frac{1}{H_{P|0}^{'n-1} - z} \right)^{1/2} \right\|_{\mathcal{F}_{|0}^n} + 2 \left\| B_{|0}^{n-1} \left(\frac{1}{H_{P|0}^{'n-1} - z} \right)^{1/2} \right\|_{\mathcal{F}_{|0}^n} + \quad (37)$$

$$+ 2 \left\| (P - P^f) \left(\frac{1}{H_{P|0}^{'n-1} - z} \right)^{1/2} \right\|_{\mathcal{F}_{|0}^n} \right]. \quad (38)$$

Notice that the standard inequalities in (22) yield

$$\begin{aligned} \|B_{|m}^n \psi\| &\leq |g| C \left(\frac{1}{\sigma_m} - \frac{1}{\sigma_n} \right)^{1/2} \|(H^f|_m^n)^{1/2} \psi\|, \\ \|B_{|m}^{*n} \psi\| &\leq |g| C \left(\left(\frac{1}{\sigma_m} - \frac{1}{\sigma_n} \right)^{1/2} \|(H^f|_m^n)^{1/2} \psi\| + (\ln \sigma_n - \ln \sigma_m)^{1/2} \|\psi\| \right) \end{aligned} \quad (39)$$

for all ψ in the domain of $H_{P,0}^{1/2}$. Then expression (35)-(38) can be controlled as follows:

1. We estimate

$$\left\| B_{n-1}^n \left(\frac{1}{H_{P|0}^{'n-1} - z} \right)^{1/2} \right\|_{\mathcal{F}_{|0}^n} \leq |g| C \left(\frac{\beta - 1}{\sigma_n} \right)^{1/2} \left\| (H^f|_{n-1}^n)^{1/2} \left(\frac{1}{H_{P|0}^{'n-1} - z} \right)^{1/2} \right\|_{\mathcal{F}_{|0}^n}. \quad (40)$$

Furthermore, since $H^f|_{n-1}^n$ and $H_{P|0}^{'n-1}$ commute, we have that

$$\begin{aligned} &\left\| (H^f|_{n-1}^n)^{1/2} \left(\frac{1}{H_{P|0}^{'n-1} - z} \right)^{1/2} \right\|_{\mathcal{F}_{|0}^n} \\ &\leq \theta^{-1/2} \left\| \left(\frac{\theta H^f|_{n-1}^n}{H_{P|0}^{'n-1} - \theta H^f|_{n-1}^n - E_{P|0}^{'n-1} + \theta H^f|_{n-1}^n + E_{P|0}^{'n-1} - z} \right)^{1/2} \right\|_{\mathcal{F}_{|0}^n} \\ &\leq \theta^{-1/2} \left\| \left(\frac{\theta H^f|_{n-1}^n}{\xi_{n-1} - \xi_n + \theta H^f|_{n-1}^n} \right)^{1/2} \right\|_{\mathcal{F}_{|0}^n} \leq \theta^{-1/2} \end{aligned} \quad (41)$$

for, e.g. $\theta = \frac{1}{16}$. This is true because of Lemma 4.3, the constraints on z given in (32), and the bound $\Delta \xi_n = \xi_{n-1} - \xi_n > 0$ (see Definition 4.1).

2. Next we consider the bounds

$$\left\| B_{|0}^{n-1} \left(\frac{1}{H_{P|0}^{'n-1} - z} \right)^{1/2} \right\|_{\mathcal{F}_{|0}^n} \leq |g| C \left\| (H^f|_0^{n-1})^{1/2} \left(\frac{1}{H_{P|0}^{'n-1} - z} \right)^{1/2} \right\|_{\mathcal{F}_{|0}^n}, \quad (42)$$

and

$$\begin{aligned} \left\| B^*|_0^{n-1} \left(\frac{1}{H'_{P|_0}|^{n-1} - z} \right)^{1/2} \right\|_{\mathcal{F}|_0^n} &\leq |g|C \left(\left\| (H^f|_0^{n-1})^{1/2} \left(\frac{1}{H'_{P|_0}|^{n-1} - z} \right)^{1/2} \right\|_{\mathcal{F}|_0^n} \right. \\ &\quad \left. + (\ln \beta^{n-1})^{1/2} \left\| \left(\frac{1}{H'_{P|_0}|^{n-1} - z} \right)^{1/2} \right\|_{\mathcal{F}|_0^n} \right). \end{aligned} \quad (43)$$

Terms including $H^f|_0^{n-1}$ or $(P - P^f)$ can be estimated as follows:

$$\left\| (H^f|_0^{n-1})^{1/2} \left(\frac{1}{H'_{P|_0}|^{n-1} - z} \right)^{1/2} \right\|_{\mathcal{F}|_0^n} \leq \left\| H_{P,0}^{1/2} \left(\frac{1}{H'_{P|_0}|^{n-1} - z} \right)^{1/2} \right\|_{\mathcal{F}|_0^n}, \quad (44)$$

$$\left\| (P - P^f) \left(\frac{1}{H'_{P|_0}|^{n-1} - z} \right)^{1/2} \right\|_{\mathcal{F}|_0^n} \leq \sqrt{2} \left\| H_{P,0}^{1/2} \left(\frac{1}{H'_{P|_0}|^{n-1} - z} \right)^{1/2} \right\|_{\mathcal{F}|_0^n}. \quad (45)$$

In order to estimate the right-hand side in (44) and (45), we observe that the standard inequalities (39) readily imply that there exists a n -independent finite constant c_{uv} such that, for $|g| \leq 1$ and $|g| < \frac{1}{c_{uv}}$, $\psi \in D(H_{P,0}^{1/2})$ and $n \in \mathbb{N}$, it holds

$$\langle \psi, H_{P,0} \psi \rangle \leq \frac{1}{1 - |g|c_{uv}} \left[\langle \psi, H'_{P|_0} \psi \rangle + g^2 c_{uv}^2 \ln \sigma_n \langle \psi, \psi \rangle \right]. \quad (46)$$

Consequently, for $|g|$ sufficiently small, we can estimate

$$\left\| H_{P,0}^{1/2} \left(\frac{1}{H'_{P|_0}|^{n-1} - z} \right)^{1/2} \right\|_{\mathcal{F}|_0^n}^2 \leq C \sup_{\|\psi\|=1} \left\langle \psi, \left[1 + (|z| + |g| \ln \sigma_n) \left| \frac{1}{H'_{P|_0}|^{n-1} - z} \right| \right] \psi \right\rangle \quad (47)$$

where $\psi \in \mathcal{F}|_0^n$. Moreover, the right-hand side of

$$|z| \leq |E'_{P|_0}|^{n-1} - z| + |E'_{P|_0}|^{n-1}|$$

is uniformly bounded because, first, $|E'_{P|_0}|^{n-1} - z| \leq \xi_{n-1} \leq \frac{1}{2}\kappa$, and, second, $|E'_{P|_0}|^{n-1}| \leq C_{E'}$ by assumption. Hence, we get

$$\left\| H_{P,0}^{1/2} \left(\frac{1}{H'_{P|_0}|^{n-1} - z} \right)^{1/2} \right\|_{\mathcal{F}|_0^n}^2 \leq C \left(1 + (1 + |g| \ln \sigma_n) \left\| \left(\frac{1}{H'_{P|_0}|^{n-1} - z} \right)^{1/2} \right\|_{\mathcal{F}|_0^n}^2 \right). \quad (48)$$

Finally, the remaining norm in (48) can be controlled by

$$\left\| \left(\frac{1}{H'_{P|_0}|^{n-1} - z} \right)^{1/2} \right\|_{\mathcal{F}|_0^n}^2 \leq \max \left\{ \frac{1}{|E'_{P|_0}|^{n-1} - z|}, \frac{1}{\text{Gap}(H'_{P|_0}|^{n-1} \upharpoonright \mathcal{F}|_0^n) - |E'_{P|_0}|^{n-1} - z|} \right\} \leq \frac{C}{\Delta \xi_n} \quad (49)$$

which is due to Lemma 4.3 and the domain of z given in (32).

We recall that by Definition 4.1 the sequence $(\Delta\xi_n)_{n \in \mathbb{N}}$ tends to zero, which is a necessary ingredient in the induction scheme in the proof of Theorem 2.1. Hence, the terms proportional to $(\Delta\xi_n)^{-1/2}$ must be treated cautiously. It turns out that the sum of the terms in (35)-(38) is bounded by

$$O\left(|g|\left(\frac{(\beta-1)}{\sigma_n \Delta\xi_n}\right)^{1/2}\right) + O\left(|g|\left(\frac{(\beta-1) \ln \beta^{n-1}}{\sigma_n \Delta\xi_n}\right)^{1/2}\right) \leq |g|^{1/2} C \left(\frac{(\beta-1)^2 n}{\beta^n \Delta\xi_n}\right)^{1/2} \quad (50)$$

for $|g| \leq (\beta-1)$; see (25). This dictates the choice $\Delta\xi_n := \frac{(\beta-1)^2}{2\beta} \frac{n}{\beta^n}$ made in Definition 4.1. Hence, for all $n \in \mathbb{N}$ we get

$$\left\| \left(\frac{1}{H'_{p|_0|^{n-1}} - z} \right)^{1/2} \Delta H'_{p|_0|^{n-1}} \left(\frac{1}{H'_{p|_0|^{n-1}} - z} \right)^{1/2} \right\|_{\mathcal{F}_0^n} \leq |g|^{1/2} C \left(\frac{(\beta-1)^2 n}{\beta^n \Delta\xi_n} \right)^{1/2} \leq |g|^{1/2} C. \quad (51)$$

Therefore, (33) holds for $|g|$ sufficiently small which proves the claim. \square

Definition 4.6. For $n \in \mathbb{N}$ we define the contour

$$\Gamma_n := \left\{ z \in \mathbb{C} \mid |E'_{p|_0|^{n-1}} - z| = \frac{1}{2} \xi_n \right\}.$$

The bound in (50) was delicate because the outer boundary of the domain of z might be close to the spectrum. However, when considering z being further away from the spectrum we get a much better estimate:

Corollary 4.7. Let g, β fulfill the conditions of Lemma 4.5 and $z \in \Gamma_n$ or $z = E'_{p|_0|_0^n} + i\lambda$ with $\lambda \in \mathbb{R}, |\lambda| = 1$ for $n \in \mathbb{N}$. The following estimates hold true

$$\left\| \left(\frac{1}{H'_{p|_0|^{n-1}} - z} \right)^{1/2} \Delta H'_{p|_0|^{n-1}} \left(\frac{1}{H'_{p|_0|^{n-1}} - z} \right)^{1/2} \right\|_{\mathcal{F}_0^n} \leq C |g| \left(\frac{(\beta-1)n}{\beta^n} \right)^{1/2}, \quad (52)$$

$$\left\| \frac{1}{H'_{p|_0|_0^n} - z} - \frac{1}{H'_{p|_0|^{n-1}} - z} \right\|_{\mathcal{F}_0^n} \leq C |g| \left(\frac{(\beta-1)n}{\beta^n} \right)^{1/2}. \quad (53)$$

Proof. It is enough to notice that in the estimate of the left-hand side of (52) one can just replace $\Delta\xi_n$ in (50) by a constant and use that $1 < \beta < 2$, see (25). For $|g|$ small enough, the inequality in (53) follows from (52). \square

With these lemmas at hand we prove the induction step for the removal of the ultraviolet cut-off.

Theorem 4.8. Let g, β fulfill the assumptions of Lemma 4.5. Then for $|g|$ sufficiently small the following holds true for all $n \in \mathbb{N}$:

(i) $E'_{p|_0|_0^n} := \inf \text{Spec} \left(H'_{p|_0|_0^n} \upharpoonright \mathcal{F}_0^n \right)$ is a non-degenerate eigenvalue of $H'_{p|_0|_0^n} \upharpoonright \mathcal{F}_0^n$.

(ii) $\text{Gap} \left(H'_{p|_0|_0^n} \upharpoonright \mathcal{F}_0^n \right) \geq \xi_n$.

(iii) The vectors

$$\begin{aligned}\Psi'_{p|0}{}^0 &:= \Omega, \\ \Psi'_{p|0}{}^j &:= Q'_{p|0}{}^j \Psi'_{p|0}{}^{j-1}, \quad Q'_{p|0}{}^j := -\frac{1}{2\pi i} \oint_{\Gamma_j} \frac{dz}{H'_{p,j} - z}, \quad j \geq 1,\end{aligned}\tag{54}$$

are well-defined and $\Psi'_{p|0}{}^n$ is the unique ground state of $H'_{p|0}{}^n \upharpoonright \mathcal{F}|_0^n$.

(iv) The following holds:

$$\|\Psi'_{p|0}{}^n - \Psi'_{p|0}{}^{n-1}\| \leq C|g| \left(\frac{(\beta-1)n}{\beta^n} \right)^{1/2}, \tag{55}$$

$$\|\Psi'_{p|0}{}^n\| \geq C_{\Psi'}, \tag{56}$$

where $0 < C_{\Psi'} < 1$.

(v) $E'_{p|0}{}^n$ is analytic in P for all $n \in \mathbb{N}$ and the following bounds hold true

$$|E'_{p|0}{}^n - E'_{p|0}{}^{n-1}| \leq C|g|^2 \frac{(\beta-1)n}{\beta^n}, \quad |E'_{p|0}{}^n| < C_{E'} \left(> \frac{P^2}{2} \right), \tag{57}$$

$$|\nabla E'_{p|0}{}^n - \nabla E'_{p|0}{}^{n-1}| \leq C|g|^2 \frac{(\beta-1)n}{\beta^n}, \quad |\nabla E'_{p|0}{}^n| \leq C_{\nabla E} \left(= \frac{3}{4} \right), \tag{58}$$

where $E'_{p|0}{}^0 \equiv \frac{P^2}{2}$ and $\nabla E'_{p|0}{}^0 \equiv P$.

Proof. We prove this by induction: Statements (i)-(v) for $(n-1)$ will be referred to as assumptions A(i)-A(v) while the same statements for n are claims C(i)-C(v). For $n = 1$ the claims can be verified by direct computation and by using Lemma 4.5. Let $n > 1$ and suppose A(i)-A(v) hold.

1. Because of A(i), A(ii), and A(v) Lemma 4.3 states that

$$\text{Gap}(H'_{p|0}{}^{n-1} \upharpoonright \mathcal{F}|_0^n) \geq \xi_{n-1}.$$

Lemma 4.5 ensures that the resolvent $(H'_{p|0}{}^n - z)^{-1}$ is well-defined for $\frac{1}{2}\xi_n \leq |E'_{p|0}{}^{n-1} - z| \leq \xi_n$.

2. Hence, Kato's theorem yields claims C(i) and C(iii). As a consequence, the spectrum of $H'_{p|0}{}^n \upharpoonright \mathcal{F}|_0^n$ is contained in $\{E'_{p|0}{}^n\} \cup (E'_{p|0}{}^{n-1} + \xi_n, \infty)$ because $E'_{p|0}{}^n \leq E'_{p|0}{}^{n-1}$ by (iii) of Corollary 5.4 for $m = 0$, which proves claim C(ii).

3. Next, we prove C(iv). By A(iii) we have

$$\|\Psi'_{p|0}{}^n - \Psi'_{p|0}{}^{n-1}\| \leq \|(Q'_n - Q'_{n-1})\Psi'_{p|0}{}^{n-1}\| = O\left(|g| \left(\frac{(\beta-1)n}{\beta^n} \right)^{1/2}\right) \tag{59}$$

where we have used Lemma 4.7 and that $\|\Psi'_{p|0}{}^{n-1}\| \leq 1$ holds by construction. Furthermore, starting from the identity

$$\|\Psi'_{p|0}{}^n\|^2 = \|\Psi'_{p|0}{}^{n-1}\|^2 + \|\Psi'_{p|0}{}^n - \Psi'_{p|0}{}^{n-1}\|^2 + 2 \text{Re} \langle \Psi'_{p|0}{}^{n-1}, \Psi'_{p|0}{}^n - \Psi'_{p|0}{}^{n-1} \rangle \tag{60}$$

we conclude that

$$\|\Psi'_{P|_0}\|^2 - \|\Psi'_{P|_0}\|^{n-1} = O\left(|g|^2 \frac{(\beta-1)n}{\beta^n}\right). \quad (61)$$

Finally, since $\|\Psi'_{P|_0}\| = 1$ by definition,

$$\|\Psi'_{P|_0}\|^2 \geq 1 - \sum_{j=1}^n \left| \|\Psi'_{P|_0}\|^j - \|\Psi'_{P|_0}\|^{j-1} \right| \geq 1 - C|g|^2 \sum_{j=0}^n \frac{(\beta-1)j}{\beta^j} \geq 1 - O(|g|) \geq C_{\Psi'} > 0$$

for some positive constant $C_{\Psi'}$, and $|g|$ sufficiently small and subject to the constraint $|g| \leq (\beta-1)$; see (25).

4. In order to prove $C(v)$, first by using (52) and (56) we can estimate the energy shift as follows

$$|E'_{P|_0} - E'_{P|_0}^{n-1}| = \left| \frac{\langle \Psi'_{P|_0}, \Delta H'_{P|_{n-1}} \Psi'_{P|_0} \rangle}{\langle \Psi'_{P|_0}, \Psi'_{P|_0} \rangle} \right| = O\left(|g|^2 \frac{(\beta-1)n}{\beta^n}\right)$$

This readily implies

$$|E'_{P|_0}| \leq \frac{P^2}{2} + C|g|^2 \sum_{j=0}^n \frac{(\beta-1)j}{\beta^j} \leq C_{E'} \quad (62)$$

for some constant $C_{E'}$.

Since $(H'_P|_0)_{|P| \leq P_{\max}}$ is an analytic family of type A and $E'_{P|_0}$ is an isolated eigenvalue, $E'_{P|_0}$ is an analytic function of P and

$$\nabla E'_{P|_0} = P - \langle [P^f + B|_0^n + B^*|_0^n] \rangle_{\Psi'_{P|_0}}. \quad (63)$$

By using equations (40), (41), (42), (45), (46) for $z \in \Gamma_n$ (see Definition 4.6), and (59), for $|g|$ sufficiently small one can easily prove that

$$\begin{aligned} \nabla E'_{P|_0} - \nabla E'_{P|_0}^{n-1} &= -\langle [B|_{n-1}^n + B^*|_{n-1}^n] \rangle_{\Psi'_{P|_0}} \\ &\quad + \langle [P - P^f + B|_0^{n-1} + B^*|_0^{n-1}] \rangle_{\Psi'_{P|_0}} - \langle [P - P^f + B|_0^{n-1} + B^*|_0^{n-1}] \rangle_{\Psi'_{P|_0}^{n-1}} \\ &= O\left(|g|^2 \frac{(\beta-1)n}{\beta^n}\right) \end{aligned}$$

and finally the bound $|\nabla E'_{P|_0}| \leq \frac{3}{4} = C_{\nabla E}$. □

We can now prove the first main result.

Proof of Theorem 2.1 in Section 2.

(i) Recall that $\Psi_P|_0^n := e^{-T|_0^n} \Psi'_P|_0^n$. By unitarity of the Gross transformation

$$\begin{aligned} \|\Psi_P|_0^n - \Psi_P|_0^{n-1}\| &= \|\Psi'_P|_0^n - e^{T|_0^{n-1}} \Psi'_P|_0^{n-1}\| \\ &\leq \|(e^{T|_0^{n-1}} - 1) \Psi'_P|_0^{n-1}\| + \|\Psi'_P|_0^n - \Psi'_P|_0^{n-1}\| \end{aligned}$$

holds. The convergence of $(\Psi'_P|_0^n)_{n \in \mathbb{N}}$ to a non-zero vector (see Theorem 4.8) and

$$\begin{aligned} \|(e^{T|_0^{n-1}} - 1) \Psi'_P|_0^{n-1}\| &\leq \int_0^1 d\lambda \|e^{\lambda T|_0^{n-1}} T|_0^{n-1} \Psi'_P|_0^{n-1}\| \\ &\leq \|T|_0^{n-1} \Psi'_P|_0^{n-1}\| \xrightarrow{n \rightarrow \infty} 0 \end{aligned}$$

imply the claim.

(ii) Again the unitarity of the Gross transformation and (5) implies

$$E_P|_0^n - V_{\text{self}}|_0^n := \inf \text{Spec}(H_P|_0^n \upharpoonright \mathcal{F}|_0^n) - V_{\text{self}}|_0^n = E'_P|_0^n. \quad (64)$$

Since the right-hand side of (57) in Theorem 4.8 is summable, the sequence $(E'_P|_0^n)$ is convergent.

(iii) By Corollary 4.7 the resolvent $(H'_P|_0^n - z)^{-1}$, for $z = E'_P|_0^n + i\lambda$, $\lambda \in \mathbb{R}$ and $|\text{Im } \lambda| = 1$, converges as $n \rightarrow \infty$. Furthermore, for every n the range of $(H'_P|_0^n - z)^{-1}$ is given by $D(H_{P,0})$ which is dense in \mathcal{F} . Hence, the Trotter-Kato Theorem [RS81, Theorem VIII.22] ensures the existence of a limiting self-adjoint Hamiltonian $H'_P|_0^\infty$ on \mathcal{F} . Because of the unitarity of the Gross transformation, the family of Hamiltonians $H_P|_0^n - V_{\text{self}}|_0^n$, $n \in \mathbb{N}$, converges to $H_P|_0^\infty := e^{-T|_0^\infty} H'_P|_0^\infty e^{T|_0^\infty}$ in the norm resolvent sense as $n \rightarrow \infty$.

(iv) By (iii), $\Psi_P|_0^\infty$ is a ground state of $H_P|_0^\infty$. Moreover, Theorem 4.8 ensures

$$\text{Spec}((H'_P|_0^n - E'_P|_0^n) \upharpoonright \mathcal{F}|_0^n) \subset \{0\} \cup (\xi_n, \infty).$$

Since $\xi_n \geq \frac{1}{16}\kappa$ the set $(-\infty, 0) \cup (0, \frac{1}{16}\kappa)$ is not part of the spectrum of $(H'_P|_0^n - E'_P|_0^n) \upharpoonright \mathcal{F}|_0^n$ for any $n \in \mathbb{N}$. As the spectrum cannot suddenly expand in the limit [RS81, Theorem VIII.24], this proves the claimed gap bound. The gap bound and the resolvent convergence imply that $E'_P|_0^\infty$ is a non-degenerate eigenvalue.

□

5 Ground States of the Gross Transformed Hamiltonians $H'_P|_m^\infty$ for $m \in \mathbb{N}$

So far we have studied the Gross transformed Hamiltonian $H'_P|_0^n$ for an arbitrary large n . In the following we want to add interaction slices below the frequency κ . As a preparation for this we state some important properties of the Hamiltonian

$$H'_P|_m^n := H'_P|_0^n + g\Phi|_m^0$$

for any $m \in \mathbb{N} \cup \{\infty\}$ and $n \in \mathbb{N}$. Note that for all such cut-offs the operator $H'_P|_m^n$ is a Kato small perturbation of $H_{P,0}$ and therefore self-adjoint on $D(H_{P,0})$. We collect these facts including the limiting case $n \rightarrow \infty$ in the next lemma.

Remark 5.1. In this section we implicitly assume the constraints $|P| < P_{\max}$ and $1 < \kappa < 2$. Furthermore, g and β are such that all the results of Section 4 hold true.

Lemma 5.2. Let $|g|$ be sufficiently small. For $n \in \mathbb{N}$, $m \in \mathbb{N} \cup \{\infty\}$ there exists $\lambda \in \mathbb{R}$ such the operator

$$\frac{1}{H'_{P|m} - E'_{P|0} \pm i\lambda}$$

has range $D(H_{P,0})$ and converges in norm as $n \rightarrow \infty$. Therefore, the sequence of operators $H'_{P|m}$, $n \in \mathbb{N}$, converges to a self-adjoint operator acting on \mathcal{F} in the norm resolvent sense.

Proof. Let $m \in \mathbb{N} \cup \{\infty\}$. The only non-straightforward case is $n \rightarrow \infty$. First, we show the validity of the Neumann expansion

$$\frac{1}{H'_{P|m} - E'_{P|0} \pm i\lambda} = \frac{1}{H'_{P|0} + g\Phi|_m^0 - E'_{P|0} \pm i\lambda} = R_n \sum_{j=0}^{\infty} (S R_n)^j \quad (65)$$

for

$$R_n := \frac{1}{H'_{P|0} - E'_{P|0} \pm i\lambda} \quad \text{and} \quad S = -g\Phi|_m^0.$$

With the standard inequalities (22) we estimate

$$\|S R_n\| \leq C|g| \left\| (H^f|_m^0)^{1/2} \left(\frac{1}{H'_{P|0} - E'_{P|0} \pm i\lambda} \right)^{1/2} \right\| \left\| \left(\frac{1}{|\lambda|} \right)^{1/2} + C|g| \frac{1}{|\lambda|} \right\|. \quad (66)$$

Fix a θ' such that $1 - \theta' - C_{\nabla E} > 0$. From an analogous computation as conducted in the proof of Lemma 4.3 one finds

$$\inf_{\|\psi\|=1} \langle \psi, (H'_{P|0} - \theta' H^f|_m^0 - E'_{P|0}) \psi \rangle \geq 0$$

where the infimum is taken over $\psi \in D(H_{P,0})$. Consequently, we get that

$$\left\| \left(\frac{\theta' H^f|_m^0}{H'_{P|0} - \theta' H^f|_m^0 - E'_{P|0} + \theta' H^f|_m^0 \pm i\lambda} \right)^{1/2} \right\|^2 \leq \frac{1}{|\lambda|}$$

holds because $H^f|_m^0$ and $H'_{P|0}$ commute. For $|\lambda|$ sufficiently large this gives

$$(66) \leq \frac{|g|C\theta'^{-1/2} + |g|C}{|\lambda|} < 1 \quad (67)$$

so that the Neumann expansion in (65) is well-defined for all $n \in \mathbb{N}$. Moreover, the limit of (65) for $n \rightarrow \infty$ exists because:

1. The sequence $(R_n)_{n \in \mathbb{N}}$, converges in norm; see Theorem 2.1
2. $\|R_l S\|, \|S R_l\| < 1$ for all $l \in \mathbb{N}$, see (67)

3. For any $j \geq 1$ we have

$$\|R_l(S R_l)^{j+1} - R_n(S R_n)^{j+1}\| \leq \|S R_l\| \|R_l(S R_l)^j - R_n(S R_n)^j\| + \|R_n S\|^{j+1} \|R_l - R_n\|.$$

For all $n \in \mathbb{N}$ the range of the resolvent $(H'_p|_m^n - E'_p|_0^n \pm i\lambda)^{-1}$ equals $D(H_{p,0})$ and therefore it is dense. Finally the Trotter-Kato Theorem [RS81, Theorem VIII.22] ensures the existence of a self-adjoint limiting operator $H'_p|_m^\infty$ bounded from below. \square

For the Hamiltonian $H'_p|_m^n$, where the infrared cut-off τ_m is arbitrarily small but strictly larger than zero, we construct the corresponding ground state $\Psi'_p|_m^n$. For this construction we introduce a new parameter ζ and provide necessary constraints on the infrared scaling parameter γ depending on the coupling constant g .

Definition 5.3. *We consider an infrared scaling parameter γ that obeys*

$$0 < \gamma < \frac{1}{2}, \quad |g| \leq \gamma^2, \quad \sum_{j=1}^{\infty} \gamma^{\frac{j}{4}} (1+j) \leq \frac{1}{2}. \quad (68)$$

Furthermore, we fix the auxiliary constant $0 < \zeta < \frac{1}{16}$ such that

$$1 - \theta - C_{\nabla E} \geq 2\zeta$$

where $0 < \theta < \frac{1}{8}$ and $C_{\nabla E} = \frac{3}{4}$.

As we shall see later, the upper bound on ζ is constrained by the ultraviolet gap estimate; see (iv) in Theorem 2.1.

In the iterative construction of the ground state we use Corollary 5.4 below that relies on Lemma 3.2 and on Theorem 3.1 for statements (i),(ii). The estimate in (iii) is based on a simple variational argument.

Corollary 5.4. *Let $|g|$ be sufficiently small. For all $n, m \in \mathbb{N}$ the following holds true:*

(i) $-|g|c_b \leq E'_p|_m^n \leq \frac{1}{2}P^2$, where c_b is the constant introduced in Lemma 3.2.

(ii) There is a $g_{\max} > 0$ such that for $0 \leq |g| < g_{\max}$ and all $k \in \mathbb{R}^3$

$$E'_{p-k}|_m^n - E'_p|_m^n \geq -C_{\nabla E}|k|. \quad (69)$$

(iii) Assume that $E'_p|_m^{n+1}$, $E'_p|_{m+1}^n$, and $E'_p|_m^n$ are eigenvalues of $H'_p|_m^{n+1} \upharpoonright \mathcal{F}_m^{n+1}$, $H'_p|_{m+1}^n \upharpoonright \mathcal{F}_{m+1}^n$, and $H'_p|_m^n \upharpoonright \mathcal{F}_m^n$, respectively; then $E'_p|_m^{n+1}, E'_p|_{m+1}^n \leq E'_p|_m^n$.

Proof. See Appendix A. \square

Lemma 5.5. *Let $|g|$ be sufficiently small and $n \in \mathbb{N} \cup \{\infty\}$. For an integer $m \geq 1$, assume:*

(i) $E'_p|_{m-1}^n$ is the non-degenerate eigenvalue of $H'_p|_{m-1}^n \upharpoonright \mathcal{F}_{m-1}^n$ with eigenvector $\Psi'_p|_{m-1}^n$.

(ii) $\text{Gap}(H'_p|_{m-1}^n \upharpoonright \mathcal{F}_{m-1}^n) \geq \zeta\tau_{m-1}$.

This implies that $E'_{P|m-1}|^n$ is also the non-degenerate ground state energy of $H'_{P|m-1}|^n \upharpoonright \mathcal{F}|_m^n$ with eigenvector $\Psi'_{P|m-1}|^n \otimes \Omega$. Furthermore, it holds:

$$\begin{aligned} \text{Gap}(H'_{P|m-1}|^n \upharpoonright \mathcal{F}|_m^n) &\geq \inf_{\mathcal{F}|_m^n \ni \psi \perp \Psi'_{P|m-1}|^n \otimes \Omega} \langle H'_{P|m-1}|^n - \theta H^f|_m^{m-1} - E'_{P|m-1}|^n \rangle_\psi \\ &\geq 2\zeta\tau_m \end{aligned} \quad (70)$$

where the infimum is taken over $\psi \in D(H_{P,0})$.

Proof. Mimicking the steps in the proof Lemma 4.3 and the inequality in (69) we get the bound

$$\inf_{\mathcal{F}|_m^n \ni \psi \perp \Psi'_{P|m-1}|^n \otimes \Omega} \langle H'_{P|0}|^n + g\Phi|_{m-1}^0 - \theta H^f|_m^{m-1} - E'_{P|m-1}|^n \rangle_\psi \geq (1 - \theta - C_{\nabla E})\tau_m \geq 2\zeta\tau_m.$$

This gives the estimate

$$\text{Gap}(H'_{P|m-1}|^n \upharpoonright \mathcal{F}|_m^n) = \text{Gap}((H'_{P|0}|^n + g\Phi|_{m-1}^0) \upharpoonright \mathcal{F}|_m^n) \geq \min\{\zeta\tau_{m-1}, 2\zeta\tau_m\} = 2\zeta\tau_m$$

where in the last step we have used that $\gamma < \frac{1}{2}$; see (68). This proves the claim for any finite n, m . But the resolvent convergence proved in Lemma 5.2 ensures that the statements remain true in the limit $n \rightarrow \infty$ as the spectrum cannot suddenly expand in the limit [RS81, Theorem VIII.24]. \square

Lemma 5.6. *For $n \in \mathbb{N} \cup \{\infty\}$ and $m \geq 1$ there is a $g_{\max} > 0$ such that, for $|g| < g_{\max}$ and γ fulfilling the constraints in (68), the assumptions of Lemma 5.5 imply that the resolvent*

$$\frac{1}{H'_{P|m}|^n - z},$$

restricted to $\mathcal{F}|_m^n$, is well-defined in the domain

$$\frac{1}{4}\zeta\tau_m \leq |E'_{P|m-1}|^n - z| \leq \zeta\tau_m. \quad (71)$$

Proof. It is sufficient to show that

$$\left\| \left(\frac{1}{H'_{P|m-1}|^n - z} \right)^{1/2} g\Phi|_m^{m-1} \left(\frac{1}{H'_{P|m-1}|^n - z} \right)^{1/2} \right\|_{\mathcal{F}|_m^n} \quad (72)$$

is less than one for all z in the given domain. For g sufficiently small this is true because:

1. By standard inequalities in (22) the estimate

$$\left\| g\Phi|_m^{m-1} \left(\frac{1}{H'_{P|m-1}|^n - z} \right)^{1/2} \right\|_{\mathcal{F}|_m^n} \leq |g|C((1 - \gamma)\tau_{m-1})^{1/2} \left\| (H^f|_m^{m-1})^{1/2} \left(\frac{1}{H'_{P|m-1}|^n - z} \right)^{1/2} \right\|_{\mathcal{F}|_m^n} \quad (73)$$

holds true. Since $H^f|_m^{m-1}$ commutes with $H'_{P|m-1}|^n$ and using (70), the spectral theorem yields

$$\left\| (H^f|_m^{m-1})^{1/2} \left(\frac{1}{H'_{P|m-1}|^n - z} \right)^{1/2} \right\|_{\mathcal{F}|_m^n} \leq C. \quad (74)$$

2. Using Lemma 5.5 we get

$$\left\| \left(\frac{1}{H'_P|_{m-1}^n - z} \right)^{1/2} \right\|_{\mathcal{F}_m^n}^2 \leq \max \left\{ \frac{1}{\frac{1}{4}\zeta\tau_m}, \frac{1}{\zeta\tau_m} \right\} \leq \frac{4}{\zeta\tau_m}. \quad (75)$$

Combining (73), (74), and (75) we find

$$(72) \leq C|g| \left(\frac{\tau_{m-1}}{\tau_m} \right)^{1/2} = C|g|\gamma^{-1/2} \leq C|g|^{3/4}.$$

where we have used the constraints in (68). This proves the claim. \square

Inside the domain where the resolvent is well-defined, let us now introduce the integration contour that is used to iteratively construct the ground state vectors in Theorem 5.8 below.

Definition 5.7. For $m \in \mathbb{N}$ we define the contour

$$\Delta_m := \left\{ z \in \mathbb{C} \mid |E'_P|_{m-1}^n - z| = \frac{1}{2}\zeta\tau_m \right\}.$$

Theorem 5.8. Let $n \in \mathbb{N} \cup \{\infty\}$ and g, γ sufficiently small such that the constraints in (68) are fulfilled. Then for all $m \geq 0$ the following holds true:

- (i) $E'_P|_m^n := \inf \text{Spec} \left(H'_P|_m^n \upharpoonright \mathcal{F}_m^n \right)$ is the non-degenerate ground state energy of $H'_P|_m^n \upharpoonright \mathcal{F}_m^n$.
- (ii) $\text{Gap} \left(H'_P|_m^n \upharpoonright \mathcal{F}_m^n \right) \geq \zeta\tau_m$.
- (iii) The vectors

$$\begin{aligned} \Psi'_P|_0^n &:= \Psi'_P|_0^n, \\ \Psi'_P|_m^n &:= \mathcal{Q}'_P|_m^n \Psi'_P|_{m-1}^n, \quad \mathcal{Q}'_P|_m^n := -\frac{1}{2\pi i} \oint_{\Delta_m} \frac{dz}{H'_P|_m^n - z}, \quad m \geq 1, \end{aligned} \quad (76)$$

are well-defined and non-zero. The vector $\Psi'_P|_m^n$ is the unique ground state of $H'_P|_m^n \upharpoonright \mathcal{F}_m^n$.

Proof. The proof is by induction and it relies on Corollary 5.4, Lemma 5.5, and Lemma 5.6. Since the rationale can be inferred from similar steps in the proof of Theorem 4.8, we do not provide the details.

The main difference with respect to Theorem 4.8 is the fact the sequence of vectors does not converge. Moreover, here we only prove that the norm of the vector $\Psi'_P|_m^n$ is nonzero for all finite m that follows from the bound $\|\Psi'_P|_m^n\| \geq C\|\Psi'_P|_{m-1}^n\|$. The same type of argument is shown for the vectors $\phi_P|_m^n$ (with n finite) in the next section. We refer the reader to equations (100)–(106). \square

An auxiliary result needed for the next section is:

Lemma 5.9. Let $|g|$ be sufficiently small. Then for all $n, m \in \mathbb{N}$

$$(i) \quad |E'_P|_{m+1}^n - E'_P|_m^n| \leq Cg^2\gamma^m \quad (77)$$

(ii)

$$|\nabla E'_P|_m^n| \leq C_{\nabla E} \quad (78)$$

hold true, where $\nabla E'_P|_m^n$ is given by

$$\nabla E'_P|_m^n = P - \langle [P^f + B|_0^n + B^*|_0^n] \rangle_{\Psi_P|_m^n}. \quad (79)$$

Proof. (i) The claim can be seen from:

- (a) The gap estimate (70) and (i) in Corollary 5.4.
- (b) The bound

$$\theta H^f|_{m+1}^m + g\Phi|_{m+1}^m + g^2 \int_{\mathcal{B}_{\tau_m} \setminus \mathcal{B}_{\tau_{m+1}}} dk \frac{\rho(k)^2}{\theta\omega(k)} \geq 0$$

which can be inferred from completion of the square.

- (c) The inequality

$$\int_{\mathcal{B}_{\tau_m} \setminus \mathcal{B}_{\tau_{m+1}}} dk \frac{\rho(k)^2}{\theta\omega(k)} \leq \frac{C}{\theta} \gamma^m.$$

- (ii) Since $(H'_P|_m^n)_{|P| \leq P_{\max}}$, is an analytic family of type A and $E'_P|_m^n$ is an isolated eigenvalue, equation (79) holds by analytic perturbation theory. Moreover, (78) follows immediately from Corollary 5.4 (ii). □

6 Ground States of the Transformed Hamiltonians $H_P^{W'}|_\infty^n$ for $n \in \mathbb{N}$

This section provides the key result for Section 7 where we remove both limits simultaneously. Here (Section 6) we generalize the strategy employed in [Piz03] to perform the limit of a vanishing infrared cut-off τ_m uniformly in the ultraviolet cut-off σ_n .

Remark 6.1. *In this section we implicitly assume the constraints $|P| < P_{\max}$ and $1 < \kappa < 2$. Furthermore, g, β , and γ are such that all the results of Sections 4 and 5 hold true.*

Preliminaries. We collect the definitions of the transformed operators and vectors, and we explain some of their properties:

Hamiltonian	Fock space
$H_P^{W'} _m^n := W_m(\nabla E'_P _m^n) H'_P _m^n W_m(\nabla E'_P _m^n)^*$	$\mathcal{F} _m^n := \mathcal{F}(L^2(\mathcal{B}_{\sigma_n} \setminus \mathcal{B}_{\tau_m}))$
$\widetilde{H}_P^{W'} _m^n := W_m(\nabla E'_P _{m-1}^n) H'_P _m^n W_m(\nabla E'_P _{m-1}^n)^*$	$\mathcal{F} _m^n$

Notice that

$$\widetilde{H}_P^{W'}|_m^n = W_m(\nabla E'_P|_{m-1}^n) W_m(\nabla E'_P|_m^n)^* H_P^{W'}|_m^n W_m(\nabla E'_P|_m^n) W_m(\nabla E'_P|_{m-1}^n)^*. \quad (80)$$

The transformation $W_m(Q)$, $Q \in \mathbb{R}^3$ and $|Q| \leq 1$, was defined in (14) and it is unitary for all finite m . For $n, m \in \mathbb{N}$ we iteratively define the vectors

$$\begin{aligned} \phi_P|_0^n &:= \frac{\Psi'_P|_0^n}{\|\Psi'_P|_0^n\|}, \\ \widetilde{\phi}_P|_m^n &:= \widetilde{Q}'_P|_m^n \phi_P|_{m-1}^n, \quad \widetilde{Q}'_P|_m^n := -\frac{1}{2\pi i} \oint_{\Delta_m} \frac{dz}{\widetilde{H}_P^{W'}|_m^n - z} \\ \phi_P|_m^n &:= W_m(\nabla E'_P|_m^n) W_m(\nabla E'_P|_{m-1}^n)^* \widetilde{\phi}_P|_m^n \end{aligned} \quad (81)$$

where the contour Δ_m was introduced in Definition 5.7. This family of vectors is well-defined because of the unitarity of the transformations W_m and of the results of Section 5. If the vectors $\phi_P|_m^n$ and $\widetilde{\phi}_P|_m^n$ are non-zero they are by construction the (unnormalized) ground states of $H_P^{W'}|_m^n$ and $\widetilde{H}_P^{W'}|_m^n$, respectively. Assuming that these vectors are non-zero we introduce the following auxiliary definitions:

$$\begin{aligned} A_{P,m}^{(n)} &:= \int dk \, k \alpha_m(\nabla E'_P|_m^n, k) [b(k) + b^*(k)], & C_{P,m}^{(k,n)} &:= \int dk \, k \alpha_m(\nabla E'_P|_m^n, k)^2, \\ C_{P,m}^{(\omega,n)} &:= \int dk \, \omega(k) \alpha_m(\nabla E'_P|_m^n, k)^2, & C_{P,m}^{(\rho,n)} &:= 2g \int dk \, \rho(k) \alpha_m(\nabla E'_P|_m^n, k). \end{aligned} \quad (82)$$

where the function

$$\alpha_m(\nabla E'_P|_m^n, k) := -g \frac{\rho(k)}{\omega(k)} \frac{\mathbb{1}_{\mathcal{B}_\kappa \setminus \mathcal{B}_{\tau_m}}(k)}{1 - \widehat{k} \cdot \nabla E'_P|_m^n}$$

was introduced in (13). Furthermore, we define

$$\begin{aligned} R_P|_m^n &:= -\nabla E'_P|_m^n \cdot (B|_0^n + B^*|_0^n) - \frac{1}{2} \left([B|_0^n, P - P^f] + [P - P^f, B^*|_0^n] + [B|_0^n, B^*|_0^n] \right), \\ \Pi_P|_m^n &:= P^f + A_{P,m}^{(n)} + B|_0^n + B^*|_0^n \end{aligned} \quad (83)$$

$$\begin{aligned} &= W_m(\nabla E'_P|_m^n) \left(P^f + B|_0^n + B^*|_0^n \right) W_m(\nabla E'_P|_m^n)^* - C_{P,m}^{(k,n)}, \\ \Gamma_P|_m^n &:= \Pi_P|_m^n - \langle \Pi_P|_m^n \rangle_{\phi_P|_m^n}, \end{aligned} \quad (84)$$

$$C_{P,m}^{(n)} := \frac{P^2}{2} - \frac{1}{2} (P - \nabla E'_P|_m^n)^2 - \nabla E'_P|_m^n \cdot C_{P,m}^{(k,n)} + C_{P,m}^{(\omega,n)} + C_{P,m}^{(\rho,n)}.$$

Using these abbreviations and a formal computation carried out in Appendix B, one can prove that the identity

$$H_P^{W'}|_m^n = \frac{1}{2} \Gamma_P|_m^n^2 + H^f - \nabla E'_P|_m^n \cdot P^f + C_{P,m}^{(n)} + R_P|_m^n \quad (85)$$

holds on $D(H_{P,0})$ for all $n, m \in \mathbb{N}$. As in [Piz03] the ‘normal ordered’ operator $\Gamma_P|_m^n$ will play a crucial role in the next steps.

Analogously, one can verify that on $D(H_{P,0})$ and for $n, m \in \mathbb{N}$ the following identity holds true:

$$\widetilde{H}_P^{W'}|_m^n = \frac{1}{2} \left(\Gamma_P|_{m-1}^n + \widetilde{A}_{P,m}^{(n)} - A_{P,m-1}^{(n)} + \widetilde{C}_{P,m}^{(k,n)} - C_{P,m-1}^{(k,n)} \right)^2 + H^f - \nabla E'_P|_{m-1}^n \cdot P^f + \widetilde{C}_{P,m}^{(n)} + R_P|_{m-1}^n; \quad (86)$$

here we have similarly introduced, for any fixed $n \in \mathbb{N}$,

$$\begin{aligned} \widetilde{A}_{P,m}^{(n)} &:= \int dk \, k \alpha_m(\nabla E_P|_{m-1}^n, k) [b(k) + b^*(k)], & \widetilde{C}_{P,m}^{(k,n)} &:= \int dk \, k \alpha_m(\nabla E'_P|_{m-1}^n, k)^2, \\ \widetilde{C}_{P,m}^{(\omega,n)} &:= \int dk \, \omega(k) \alpha_m(\nabla E_P|_{m-1}^n, k)^2, & \widetilde{C}_{P,m}^{(\rho,n)} &:= 2g \int dk \, \rho(k) \alpha_m(\nabla E_P|_{m-1}^n, k), \end{aligned} \quad (87)$$

which differ from those in (82) only in the argument of α_m . We also define

$$\begin{aligned} \widetilde{\Pi}_P|_m^n &:= P^f + \widetilde{A}_{P,m}^{(n)} + B|_0^n + B^*|_0^n \\ &= W_m(\nabla E'_P|_{m-1}^n) \left(P^f + B|_0^n + B^*|_0^n \right) W_m(\nabla E'_P|_{m-1}^n)^* - \widetilde{C}_{P,m}^{(k,n)}, \\ \widetilde{\Gamma}_P|_m^n &:= \widetilde{\Pi}_P|_m^n - \langle \widetilde{\Pi}_P|_m^n \rangle_{\widetilde{\phi}_P|_m^n}, \\ \widetilde{C}_{P,m}^{(n)} &:= \frac{P^2}{2} - \frac{1}{2} (P - \nabla E'_P|_{m-1}^n)^2 - \nabla E'_P|_{m-1}^n \cdot C_{P,m}^{(k,n)} + \widetilde{C}_{P,m}^{(\omega,n)} + \widetilde{C}_{P,m}^{(\rho,n)}. \end{aligned} \quad (88)$$

Notice that using (79) we have the following identities

$$\langle \Pi_P|_m^n \rangle_{\phi_P|_m^n} = P - \nabla E'_P|_m^n - C_{P,m}^{(k,n)}, \quad (89)$$

$$\Gamma_P|_m^n = W_m(\nabla E'_P|_m^n) \left(P^f + B|_0^n + B^*|_0^n \right) W_m(\nabla E'_P|_m^n)^* - P + \nabla E'_P|_m^n, \quad (90)$$

$$\widetilde{\Gamma}_P|_m^n = W_m(\nabla E'_P|_{m-1}^n) W_m(\nabla E'_P|_m^n)^* \Gamma_P|_m^n W_m(\nabla E'_P|_m^n) W_m(\nabla E'_P|_{m-1}^n)^*, \quad (91)$$

$$\widetilde{\Gamma}_P|_m^n - \Gamma_P|_{m-1}^n = (\nabla E'_P|_m^n - \nabla E'_P|_{m-1}^n) + (\widetilde{A}_{P,m}^{(n)} - A_{P,m-1}^{(n)}) + (\widetilde{C}_{P,m}^{(k,n)} - C_{P,m-1}^{(k,n)}). \quad (92)$$

To start with, we show that for any finite m , the vectors $\phi_P|_m^n$ and $\widetilde{\phi}_P|_m^n$ are non-zero. Namely, by starting from $\phi_P|_0^n$, we estimate the norm difference

$$\|\widetilde{\phi}_P|_m^n - \phi_P|_{m-1}^n\| = \left\| -\frac{1}{2\pi i} \oint_{\Delta_m} \frac{dz}{\widetilde{H}_P^{W'}|_m^n - z} \phi_P|_{m-1}^n - \phi_P|_{m-1}^n \right\|. \quad (93)$$

In (93) we expand the resolvent with respect to

$$\Delta \widehat{H}_P^{W'}|_m^{m-1} := \widetilde{H}_P^{W'}|_m^n - H_P^{W'}|_{m-1}^n - \widetilde{C}_{P,m}^{(n)} + C_{P,m-1}^{(n)} \quad (94)$$

$$= \frac{1}{2} \left(\widetilde{A}_{P,m}^{(n)} - A_{P,m-1}^{(n)} + \widetilde{C}_{P,m}^{(k,n)} - C_{P,m-1}^{(k,n)} \right)^2 + \quad (95)$$

$$+ \frac{1}{2} \left[\widetilde{A}_{P,m}^{(n)} - A_{P,m-1}^{(n)}, \Gamma_P|_{m-1}^n \right] + \quad (96)$$

$$+ \left(\widetilde{A}_{P,m}^{(n)} - A_{P,m-1}^{(n)} \right) \cdot \Gamma_P|_{m-1}^n + \left(\widetilde{C}_{P,m}^{(k,n)} - C_{P,m-1}^{(k,n)} \right) \cdot \Gamma_P|_{m-1}^n. \quad (97)$$

Given the form of $\Delta \widehat{H}_P^{W'}|_m^{m-1}$ it is convenient to replace the integration contour Δ_m with $\widehat{\Delta}_m$ defined below:

Definition 6.2. For $m \in \mathbb{N}$ define

$$\widehat{\Delta}_m := \left\{ z - (C_{P,m-1}^{(n)} - \widetilde{C}_{P,m}^{(n)}) \mid z \in \Delta_m \right\}.$$

In the same fashion as Theorem 5.8 we ensure the bounds

$$\frac{1}{4}\zeta\tau_m \leq |E'_P|_{m-1}^n - z + \widetilde{C}_{P,m}^{(n)} - C_{P,m-1}^{(n)}| \leq \zeta\tau_m. \quad (98)$$

for z in the original integration contour Δ_m . For this we observe that

$$|\widetilde{C}_{P,m}^{(n)} - C_{P,m-1}^{(n)}| \leq g^2 C\tau_{m-1}, \quad (99)$$

and hence, for $|g|$ sufficiently small,

$$\zeta\tau_m \geq \frac{1}{2}\zeta\tau_m + g^2 C\tau_{m-1} \geq |E'_P|_{m-1}^n - z + \widetilde{C}_{P,m}^{(n)} - C_{P,m-1}^{(n)}| \geq \frac{1}{2}\zeta\tau_m - g^2 C\tau_{m-1} \geq \frac{1}{4}\zeta\tau_m$$

where in the last step we have used the constraints in (68). The upper bound (98) follows from (99) by a similar argument. Hence, we can use the shifted contours $\widehat{\Delta}_m$ instead of Δ_m and estimate

$$\|\widetilde{\phi}_P|_m^n - \phi_P|_{m-1}^n\| \leq \left\| \frac{1}{2\pi i} \oint_{\widehat{\Delta}_m} dz \sum_{j=1}^{\infty} \left(\frac{1}{E'_P|_{m-1}^n - z} \right)^{1/2} \left(\frac{1}{H_P^{W'}|_{m-1}^n - z} \right)^{1/2} \times \right. \quad (100)$$

$$\left. \times \left[\left(\frac{1}{H_P^{W'}|_{m-1}^n - z} \right)^{1/2} (-\Delta \widehat{H}_n^{W'}|_m^{m-1}) \left(\frac{1}{H_P^{W'}|_{m-1}^n - z} \right)^{1/2} \right]^j \phi_P|_{m-1}^n \right\|$$

$$\leq C\gamma^m \sup_{z \in \widehat{\Delta}_m} \left| \frac{1}{E'_P|_{m-1}^n - z} \right|^{1/2} \left\| \left(\frac{1}{H_P^{W'}|_{m-1}^n - z} \right)^{1/2} \right\|_{\mathcal{F}_m^n} \times \quad (101)$$

$$\times \sum_{j=1}^{\infty} \left\| \left(\frac{1}{H_P^{W'}|_{m-1}^n - z} \right)^{1/2} \Delta \widehat{H}_n^{W'}|_m^{m-1} \left(\frac{1}{H_P^{W'}|_{m-1}^n - z} \right)^{1/2} \right\|_{\mathcal{F}_m^n}^{j-1} \times \quad (102)$$

$$\times \left\| \left(\frac{1}{H_P^{W'}|_{m-1}^n - z} \right)^{1/2} \Delta \widehat{H}_n^{W'}|_m^{m-1} \left(\frac{1}{H_P^{W'}|_{m-1}^n - z} \right)^{1/2} \phi_P|_{m-1}^n \right\|. \quad (103)$$

Firstly, the gap estimate in (75) immediately yields

$$\sup_{z \in \widehat{\Delta}_m} \left| \frac{1}{E'_P|_{m-1}^n - z} \right|^{1/2} \left\| \left(\frac{1}{H_P^{W'}|_{m-1}^n - z} \right)^{1/2} \right\|_{\mathcal{F}_m^n} \leq \frac{C}{\gamma^m}$$

so that (101) is bounded by a constant. Secondly, we show that the series in (102) is convergent. We remark that $(\widetilde{A}_{P,m}^{(n)} - A_{P,m-1}^{(n)})$ commutes with $W_{m-1}(\nabla E'_P|_{m-1}^n)$ so that

$$\left\| \left(\frac{1}{H_P^{W'}|_{m-1}^n - z} \right)^{1/2} (\widetilde{A}_{P,m}^{(n)} - A_{P,m-1}^{(n)}) \cdot \Gamma_P|_{m-1}^n \left(\frac{1}{H_P^{W'}|_{m-1}^n - z} \right)^{1/2} \right\|_{\mathcal{F}_m^n}$$

$$= \left\| \left(\frac{1}{H_P^{W'}|_{m-1}^n - z} \right)^{1/2} (\widetilde{A}_{P,m}^{(n)} - A_{P,m-1}^{(n)}) \cdot (P^f + B|_0^n + B^*|_0^n + \nabla E'_P|_{m-1}^n - P) \left(\frac{1}{H_P^{W'}|_{m-1}^n - z} \right)^{1/2} \right\|_{\mathcal{F}_m^n},$$

where we used again the unitarity of W_{m-1} . Since $(\widetilde{A}_{P,m}^{(n)} - A_{P,m-1}^{(n)})$ commutes with $B|_0^n$, $B^*|_0^n$ it is enough to bound

$$\begin{aligned} & \left\| \left(\frac{1}{H_{P,m-1}'|_m^n - z} \right)^{1/2} (\widetilde{A}_{P,m}^{(n)} - A_{P,m-1}^{(n)}) \cdot [P^f - P + B|_0^n] \left(\frac{1}{H_{P,m-1}'|_m^n - z} \right)^{1/2} \right\|_{\mathcal{F}_m^n} \\ & \leq C \left\| (\widetilde{A}_{P,m}^{(n)} - A_{P,m-1}^{(n)}) \cdot \left(\frac{1}{H_{P,m-1}'|_m^n - z} \right)^{1/2} \right\|_{\mathcal{F}_m^n} \times \end{aligned} \quad (104)$$

$$\times \left[\left\| H_{P,0}^{1/2} \left(\frac{1}{H_{P,m-1}'|_m^n - z} \right)^{1/2} \right\|_{\mathcal{F}_m^n} + \left\| B|_0^n \left(\frac{1}{H_{P,m-1}'|_m^n - z} \right)^{1/2} \right\|_{\mathcal{F}_m^n} \right] \quad (105)$$

The factor (104) can be bounded by $C|g|\gamma^{(m-1)/2}$, similarly to (73). Both terms in (105) can be estimated as $C|g|\gamma^{-m/2}$ using inequalities (22)-(23) and the uniform bound on $|E_{P,m}'|$ given by Corollary 5.4; see an analogous argument in (48) that exploits the bound in (46). All the remaining terms can be controlled in a similar fashion. Hence, for $|g|$ sufficiently small and γ satisfying the constraint (68), we conclude that

$$\|\widetilde{\phi}_P|_m^n\| \geq C\|\phi_P|_{m-1}^n\| \quad (106)$$

for a strictly positive constant C .

Key result. Theorem 6.3 below is the key tool needed for proving the second main result of this paper, namely that the ground states $(\phi_P|_m^n)_{m \in \mathbb{N}}$ converge to a non-zero vector. This theorem relies on several lemmas (Lemma 6.4, Lemma 6.5, and Lemma 6.6) that will be proven later on.

Recall that the symbol C denotes any universal constant. Throughout the computation it will be important to distinguish the constants C_i , $1 \leq i \leq 7$.

Theorem 6.3. *For $|g|$, γ , and ζ sufficiently small and fulfilling the constraints in Definition 5.3 the following holds true for all $n \in \mathbb{N}$, $m \geq 1$:*

- (i) $\|\phi_P|_m^n - \widetilde{\phi}_P|_m^n\| \leq m\gamma^{\frac{m}{4}}$ and $\|\widetilde{\phi}_P|_m^n - \phi_P|_{m-1}^n\| \leq \gamma^{\frac{m}{4}}$,
- (ii) $\|\phi_P|_m^n\| \geq 1 - \sum_{j=1}^m \gamma^{\frac{j}{4}}(1+j) \quad (\geq \frac{1}{2})$,
- (iii) Let $z \in \widehat{\Delta}_{m+1}$ and $\delta := \frac{1}{2}$ then

$$|g|^\delta \left| \left\langle \Gamma_P^{(i)}|_m^n \phi_P|_m^n, \frac{1}{H_{P,m-1}^{W'}|_m^n - z} \Gamma_P^{(i)}|_m^n \phi_P|_m^n \right\rangle \right| \leq \gamma^{-\frac{m}{2}}, \quad i = 1, 2, 3.$$

Proof. We prove this by induction: Statements (i)-(iii) for $(m-1)$ shall be referred to as assumptions A(i)-A(iii) while the same statements for m are referred to as claims C(i)-C(iii).

A straightforward computation yields the case $m = 1$.

Let $m \geq 2$ and suppose A(i)-A(iii) hold. We start proving claims C(i) and C(ii).

1. Due to the inequality in (101)-(103), the estimate

$$\|\widetilde{\phi}_P|_m^n - \phi_P|_{m-1}^n\| \leq C_1 \left\| \left(\frac{1}{H_P^{W'}|_{m-1}^n - z} \right)^{1/2} \Delta \widehat{H}_n^{W'}|_m^{m-1} \left(\frac{1}{H_P^{W'}|_{m-1}^n - z} \right)^{1/2} \phi_P|_{m-1}^n \right\|$$

holds true for $|g|$ sufficiently small, uniformly in n and m . Furthermore, Lemma 6.5 states that

$$\begin{aligned} & \left\| \left(\frac{1}{H_P^{W'}|_{m-1}^n - z} \right)^{1/2} \Delta \widehat{H}_n^{W'}|_m^{m-1} \left(\frac{1}{H_P^{W'}|_{m-1}^n - z} \right)^{1/2} \phi_P|_{m-1}^n \right\| \\ & \leq |g| C_2 \gamma^{\frac{m-2}{2}} \left(1 + \sum_{i=1}^3 \left| \left\langle \Gamma_P^{(i)}|_{m-1}^n \phi_P|_{m-1}^n, \frac{1}{H_P^{W'}|_{m-1}^n - z} \Gamma_P^{(i)}|_{m-1}^n \phi_P|_{m-1}^n \right\rangle \right|^{\frac{1}{2}} \right) \end{aligned}$$

which together with the induction assumption A(iii) yields

$$\|\widetilde{\phi}_P|_m^n - \phi_P|_{m-1}^n\| \leq |g| C_1 C_2 \gamma^{\frac{m-2}{2}} \left(1 + 3|g|^{-\frac{\delta}{2}} \gamma^{-\frac{m-1}{4}} \right).$$

For $|g|$ sufficiently small and γ satisfying the constraints in (68) we have

$$\|\widetilde{\phi}_P|_m^n - \phi_P|_{m-1}^n\| \leq \gamma^{\frac{m}{4}}. \quad (107)$$

Finally, from (107), A(ii) and (68) we conclude

$$\|\widetilde{\phi}_P|_m^n\| \geq \|\phi_P|_{m-1}^n\| - \|\widetilde{\phi}_P|_m^n - \phi_P|_{m-1}^n\| \geq 1 - \sum_{j=1}^{m-1} \gamma^{\frac{j}{4}} (1+j) - \gamma^{\frac{m}{4}} \geq \frac{1}{2}. \quad (108)$$

2. We observe that

$$\begin{aligned} \|\phi_P|_m^n - \widetilde{\phi}_P|_m^n\| & \leq \| [W_m(\nabla E'_P|_m^n) W_m(\nabla E'_P|_{m-1}^n)^* - \mathbb{1}_{\mathcal{F}|_m^n}] \widetilde{\phi}_P|_m^n \| \\ & \leq \left\| [W_m(\nabla E'_P|_m^n) - W_m(\nabla E'_P|_{m-1}^n)] \frac{\Psi'_P|_m^n}{\|\Psi'_P|_m^n\|} \right\| \end{aligned} \quad (109)$$

holds because the vectors $\Psi'_P|_m^n$ and $W_m(\nabla E'_P|_{m-1}^n)^* \widetilde{\phi}_P|_m^n$ are parallel and $\|\widetilde{\phi}_P|_m^n\| \leq 1$. Lemma 6.6 yields

$$(109) \leq |g| C_3 m |\ln \gamma| \left| \nabla E'_P|_m^n - \nabla E'_P|_{m-1}^n \right|. \quad (110)$$

The difference of the gradients of the ground state energies in (110) is estimated in Lemma 6.7 which states that

$$\begin{aligned} |\nabla E'_P|_m^n - \nabla E'_P|_{m-1}^n| & \leq g^2 C_4 \gamma^{\frac{m-1}{2}} + \sup_{z \in \widehat{\Delta}_m} \left\| \left(\frac{1}{H_P^{W'}|_m^n - z} \right)^{1/2} \Delta \widehat{H}_n^{W'}|_m^{m-1} \left(\frac{1}{H_P^{W'}|_m^n - z} \right)^{1/2} \phi_P|_{m-1}^n \right\| \\ & \quad + C \frac{\|\phi_P|_{m-1}^n - \widetilde{\phi}_P|_m^n\|}{\|\phi_P|_{m-1}^n\|^2 \|\widetilde{\phi}_P|_m^n\|^2}. \end{aligned}$$

Hence, using Lemma 6.5, (107), (108) as well as assumptions A(ii) and A(iii), one finds that

$$\|\phi_P|_m^n - \widetilde{\phi}_P|_m^n\| \leq |g|C_3m|\ln \gamma| \left(g^2C_4\gamma^{m-1} + |g|C_2\gamma^{\frac{m-2}{2}}(1 + 3|g|^{-\frac{\delta}{2}}\gamma^{-\frac{m-1}{4}}) + C_5\gamma^{\frac{m}{4}} \right)$$

which implies

$$\|\phi_P|_m^n - \widetilde{\phi}_P|_m^n\| \leq m\gamma^{\frac{m}{4}} \quad (111)$$

for $|g|$ sufficiently small and γ fulfilling the constraints in (68).

Estimates (107) and (111) prove C(i). C(ii) follows along the same lines as (108) using the bound in (111).

Finally, we prove claim C(iii). Let $z \in \widehat{\Delta}_{m+1}$ and $i = 1, 2, 3$. Using the unitarity of the transformations W_m we get

$$\left| \left\langle \Gamma_P^{(i)}|_m^n \phi_P|_m^n, \frac{1}{H_P^{W'}|_m^n - z} \Gamma_P^{(i)}|_m^n \phi_P|_m^n \right\rangle \right| = \left| \left\langle \widetilde{\Gamma}_P^{(i)}|_m^n \widetilde{\phi}_P|_m^n, \frac{1}{\widetilde{H}_P^{W'}|_m^n - z} \widetilde{\Gamma}_P^{(i)}|_m^n \widetilde{\phi}_P|_m^n \right\rangle \right|,$$

see identities (80)-(91). For $|g|$ sufficiently small, i.e., $|g|$ of order γ^2 , we can expand the resolvent $(\widetilde{H}_P^{W'}|_m^n - z)^{-1}$ by the same reasoning as for (100)-(103) even for $z \in \widehat{\Delta}_{m+1}$ because of the bound on the energy shifts

$$|E_P|_{m+1}^n - E_P|_m^n| \leq Cg^2\gamma^m, \quad (112)$$

given by Lemma 5.9, and because of (71). Hence, using (94) we find

$$\begin{aligned} & \left| \left\langle \widetilde{\Gamma}_P^{(i)}|_m^n \widetilde{\phi}_P|_m^n, \frac{1}{\widetilde{H}_P^{W'}|_m^n - z} \widetilde{\Gamma}_P^{(i)}|_m^n \widetilde{\phi}_P|_m^n \right\rangle \right| \\ & \leq \sum_{j=1}^{\infty} \left\| \left(\frac{1}{H_P^{W'}|_{m-1}^n - z} \right)^{1/2} [\Delta \widehat{H}_n^{W'}|_m^{m-1} + \widetilde{C}_{P,m}^{(n)} - C_{P,m-1}^{(n)}] \left(\frac{1}{H_P^{W'}|_{m-1}^n - z} \right)^{1/2} \right\|_{\mathcal{F}|_m^n}^{j-1} \times \\ & \quad \times \left\| \left(\frac{1}{H_P^{W'}|_{m-1}^n - z} \right)^{1/2} \widetilde{\Gamma}_P^{(i)}|_m^n \widetilde{\phi}_P|_m^n \right\|^2 \\ & \leq C \left\| \left(\frac{1}{H_P^{W'}|_{m-1}^n - z} \right)^{1/2} \widetilde{\Gamma}_P^{(i)}|_m^n \widetilde{\phi}_P|_m^n \right\|^2 \end{aligned} \quad (113)$$

Furthermore,

$$\left\| \left(\frac{1}{H_P^{W'}|_{m-1}^n - z} \right)^{1/2} \widetilde{\Gamma}_P^{(i)}|_m^n \widetilde{\phi}_P|_m^n \right\|^2 \leq 2 \left\| \left(\frac{1}{H_P^{W'}|_{m-1}^n - z} \right)^{1/2} \Gamma_P^{(i)}|_{m-1}^n \phi_P|_{m-1}^n \right\|^2 + \quad (114)$$

$$+ 2 \left\| \left(\frac{1}{H_P^{W'}|_{m-1}^n - z} \right)^{1/2} (\widetilde{\Gamma}_P^{(i)}|_m^n \widetilde{\phi}_P|_m^n - \Gamma_P^{(i)}|_{m-1}^n \phi_P|_{m-1}^n) \right\|^2. \quad (115)$$

Term (114): Exploiting the property

$$\langle \phi_P|_{m-1}^n, \Gamma_P|_{m-1}^n \phi_P|_{m-1}^n \rangle = 0$$

and the spectral theorem, one can show that the term on the right-hand side of (114) fulfills

$$\begin{aligned} \left\| \left(\frac{1}{H_P^{W'}|_{m-1} - z} \right)^{1/2} \Gamma_P^{(i)}|_{m-1} \phi_P|_{m-1}^n \right\|^2 &= \left\langle \Gamma_P^{(i)}|_{m-1} \phi_P|_{m-1}^n, \left| \frac{1}{H_P^{W'}|_{m-1} - z} \right| \Gamma_P^{(i)}|_{m-1} \phi_P|_{m-1}^n \right\rangle \\ &\leq C \left\langle \Gamma_P^{(i)}|_{m-1} \phi_P|_{m-1}^n, \frac{1}{H_P^{W'}|_{m-1} - z} \Gamma_P^{(i)}|_{m-1} \phi_P|_{m-1}^n \right\rangle \end{aligned} \quad (116)$$

$$\leq C \left\langle \Gamma_P^{(i)}|_{m-1} \phi_P|_{m-1}^n, \frac{1}{H_P^{W'}|_{m-1} - y} \Gamma_P^{(i)}|_{m-1} \phi_P|_{m-1}^n \right\rangle \quad (117)$$

$$+ C \frac{\sup_{y \in \widehat{\Delta}_m, z \in \widehat{\Delta}_{m+1}} |z - y|}{\text{dist}(z, \text{Spec}(H_P'|_{m-1} \upharpoonright \mathcal{F}|_{m-1}^n) \setminus \{E_P'|_{m-1}\})} \left\langle \Gamma_P|_{m-1} \phi_P|_{m-1}^n, \frac{1}{H_P'|_{m-1} - y} \Gamma_P|_{m-1} \phi_P|_{m-1}^n \right\rangle \quad (118)$$

$$\leq C_7 \left\langle \Gamma_P|_{m-1} \phi_P|_{m-1}^n, \frac{1}{H_P'|_{m-1} - y} \Gamma_P|_{m-1} \phi_P|_{m-1}^n \right\rangle. \quad (119)$$

for $y \in \widehat{\Delta}_m$ (recall that $z \in \widehat{\Delta}_{m+1}$). In passing from (116) to (117) we have used the property $\langle \Gamma_P|_{m-1} \phi_P|_{m-1}^n, \phi_P|_{m-1}^n \rangle = 0$ which implies that the vector $\Gamma_P^{(i)}|_{m-1} \phi_P|_{m-1}^n$ has spectral support (with respect to $H_P^{W'}|_{m-1}$) contained in the interval $(E_P'|_{m-1} + \zeta\tau_{m-1}, \infty)$, and hence:

a)

$$\left\langle \Gamma_P|_{m-1} \phi_P|_{m-1}^n, \left| \frac{1}{H_P'|_{m-1} - y} \right| \Gamma_P|_{m-1} \phi_P|_{m-1}^n \right\rangle \leq C \left\langle \Gamma_P|_{m-1} \phi_P|_{m-1}^n, \frac{1}{H_P'|_{m-1} - y} \Gamma_P|_{m-1} \phi_P|_{m-1}^n \right\rangle$$

b)

$$\begin{aligned} &\left\langle \Gamma_P|_{m-1} \phi_P|_{m-1}^n, \frac{1}{H_P'|_{m-1} - z} \frac{1}{H_P'|_{m-1} - y} \Gamma_P|_{m-1} \phi_P|_{m-1}^n \right\rangle \\ &\leq \frac{1}{\text{dist}(z, \text{Spec}(H_P'|_{m-1} \upharpoonright \mathcal{F}|_{m-1}^n) \setminus \{E_P'|_{m-1}\})} \left\langle \Gamma_P|_{m-1} \phi_P|_{m-1}^n, \left| \frac{1}{H_P'|_{m-1} - y} \right| \Gamma_P|_{m-1} \phi_P|_{m-1}^n \right\rangle. \end{aligned}$$

In the step from (116)-(118) we used inequality (112). Therefore, we can conclude that

$$(114) \leq C_7 \left\langle \Gamma_P|_{m-1} \phi_P|_{m-1}^n, \frac{1}{H_P'|_{m-1} - y} \Gamma_P|_{m-1} \phi_P|_{m-1}^n \right\rangle. \quad (120)$$

Term (115): We first observe that

$$(115) \leq 4 \left\| \left(\frac{1}{H_P^{W'}|_{m-1} - z} \right)^{1/2} (\widetilde{\Gamma}_P^{(i)}|_m - \Gamma_P^{(i)}|_{m-1}) \widetilde{\phi}_P|_m^n \right\|^2 + \quad (121)$$

$$+ 4 \left\| \left(\frac{1}{H_P^{W'}|_{m-1} - z} \right)^{1/2} \Gamma_P^{(i)}|_{m-1} (\widetilde{\phi}_P|_m^n - \phi_P|_{m-1}^n) \right\|^2. \quad (122)$$

In order to estimate (121) we use the identity in (92) and the following ingredients:

c) The bound on $|\nabla E'_P|_m^n - \nabla E'_P|_{m-1}^n|$ from Lemma 6.7

d) The estimate in (99), i.e. $|\widetilde{C}_{P,m}^{(k,n)} - C_{P,m-1}^{(k,n)}| \leq g^2 C \gamma^{m-1}$

e) The bound

$$\left\| \left(\frac{1}{H_P^{W'}|_{m-1}^n - z} \right)^{1/2} \int dk \, k [\alpha_m(\nabla E'_P|_{m-1}^n, k) - \alpha_{m-1}(\nabla E'_P|_{m-1}^n, k)] (b(k) + b^*(k)) \right\|_{\mathcal{F}_m^n}^2 \leq g^2 C \gamma^{m-3}.$$

Hence, we obtain

$$(121) \leq \frac{C}{\tau_{m+1}} \left[g^2 \tau_{m-1}^{1/2} + \sup_{y \in \widehat{\Delta}_m} \left\| \left(\frac{1}{H_P^{W'}|_{m-1}^n - y} \right)^{1/2} \Delta \widehat{H}_n^{W'}|_m^{m-1} \left(\frac{1}{H_P^{W'}|_{m-1}^n - y} \right)^{1/2} \phi_P|_{m-1}^n \right\| + \right. \quad (123)$$

$$\left. + \frac{\|\phi_P|_{m-1}^n - \widetilde{\phi}_P|_m^n\|^2}{\|\phi_P|_{m-1}^n\|^2 \|\widetilde{\phi}_P|_m^n\|^2} \right] + \quad (124)$$

$$+ \frac{C}{\tau_{m+1}} [g^2 C \gamma^{m-1}]^2 \quad (125)$$

$$+ g^2 C \gamma^{m-3} \quad (126)$$

where (123)-(124), (125) and (126) are related to ingredients c), d) and e) respectively.

For the remaining term (122) we use analytic perturbation theory to find

$$\begin{aligned} \sqrt{(122)} &\leq C \tau_m \sup_{y \in \widehat{\Delta}_m} \left\| \left(\frac{1}{H_P^{W'}|_{m-1}^n - z} \right)^{1/2} \Gamma_P^{(i)}|_{m-1}^n \left(\frac{1}{H_P^{W'}|_{m-1}^n - y} \right)^{1/2} \right\|_{\mathcal{F}_m^n} \times \\ &\quad \times \sum_{j=1}^{\infty} \left\| \left(\frac{1}{H_P^{W'}|_{m-1}^n - y} \right)^{1/2} \Delta \widehat{H}_n^{W'}|_m^{m-1} \left(\frac{1}{H_P^{W'}|_{m-1}^n - y} \right)^{1/2} \right\|_{\mathcal{F}_m^n}^{j-1} \times \\ &\quad \times \left\| \left(\frac{1}{H_P^{W'}|_{m-1}^n - y} \right)^{1/2} \Delta \widehat{H}_n^{W'}|_m^{m-1} \left(\frac{1}{H_P^{W'}|_{m-1}^n - y} \right)^{1/2} \phi_P|_{m-1}^n \right\| \left| \frac{1}{E'_P|_{m-1}^n - y} \right|^{1/2} \\ &\leq \frac{C}{\gamma^{\frac{1}{2}}} \frac{1}{\gamma^{\frac{m}{2}}} \sup_{y \in \widehat{\Delta}_m} \left\| \left(\frac{1}{H_P^{W'}|_{m-1}^n - y} \right)^{1/2} \Delta \widehat{H}_n^{W'}|_m^{m-1} \left(\frac{1}{H_P^{W'}|_{m-1}^n - y} \right)^{1/2} \phi_P|_{m-1}^n \right\|, \end{aligned}$$

where we have used the estimates in (101)-(103) for $y \in \widehat{\Delta}_m$, and, using the identity in (90)

$$\left\| \left(\frac{1}{H_P^{W'}|_{m-1}^n - z} \right)^{1/2} \Gamma_P^{(i)}|_{m-1}^n \left(\frac{1}{H_P^{W'}|_{m-1}^n - y} \right)^{1/2} \right\|_{\mathcal{F}_m^n} \quad (127)$$

$$\begin{aligned} &= \left\| \left(\frac{1}{H_P^{W'}|_{m-1}^n - z} \right)^{1/2} [P^f - P + \nabla E'_P|_{m-1}^n + B|_0^n + B^*|_0^n] \left(\frac{1}{H_P^{W'}|_{m-1}^n - y} \right)^{1/2} \right\|_{\mathcal{F}_m^n} \\ &\leq C \tau_m^{-1} \gamma^{-1/2}. \end{aligned} \quad (128)$$

The inequality in (128) can be derived by combining the first inequality in (39) with Lemma 3.2.

Using Lemma 6.5, Assumption A(iii), the estimates (107), (108) and the constraints (68) we get

$$(121) \leq C \left[g^4 \gamma^{-2} + g^2 \gamma^{-3} \left(1 + \gamma^{-\frac{m-1}{2}} g^{-\delta} \right) + \gamma^{-\frac{m+2}{2}} + g^4 \gamma^{m-3} + g^2 \gamma^{m-3} \right] \leq \frac{C}{\gamma^{\frac{m+2}{2}}},$$

$$(122) \leq C g^2 \gamma^{-3} (1 + \gamma^{-\frac{m-1}{2}} |g|^{-\delta}) \leq \frac{C}{\gamma^{\frac{m+2}{2}}},$$

and hence,

$$(115) \leq \frac{C_6}{\gamma^{\frac{m+2}{2}}}. \quad (129)$$

Finally, we collect inequalities (120), (129) and make use of assumption A(iii) to derive

$$|g|^\delta \left| \left\langle \Gamma_P^{(i)} |^n \phi_P |^n, \frac{1}{H_P^{W'} |^n - z} \Gamma_P^{(i)} |^n \phi_P |^n \right\rangle \right| \leq C_7 \gamma^{-\frac{m-1}{2}} + |g|^\delta \frac{C_6}{\gamma^{\frac{m+2}{2}}} \leq \gamma^{-\frac{m}{2}}$$

for γ and $|g|$ sufficiently small and fulfilling the constraints in (68). This proves claim C(iii). \square

We shall now provide the lemmas we have used.

Lemma 6.4. *Let $|g|$ be sufficiently small. For $n, m \in \mathbb{N}$ the following expectation values are uniformly bounded:*

$$\left| \langle \phi_P |^n, \Pi_P |^n \phi_P |^n \rangle \right|, \left| \langle \widetilde{\phi}_P |^n, \widetilde{\Pi}_P |^n \widetilde{\phi}_P |^n \rangle \right| \leq C,$$

Proof. We only prove the bound for the first term. The second can be bounded analogously. Let $n, m \in \mathbb{N}$. By definition of the transformations W_m and using the fact that the vectors

$$\Psi_P' |^n, \quad W_m(\nabla E_P' |^n)^* \phi_P |^n, \quad W_m(\nabla E_P' |_{m-1}^n)^* \widetilde{\phi}_P |^n$$

are parallel and their norm is less than one, we have

$$\begin{aligned} & \left| \langle \phi_P |^n, \Pi_P |^n \phi_P |^n \rangle \right| \\ & \leq C \left| \left\langle \frac{\Psi_P' |^n}{\|\Psi_P' |^n\|}, \left[P^f + B|_0^n + B^*|_0^n - C_{P,m}^{(k,n)} \right] \frac{\Psi_P' |^n}{\|\Psi_P' |^n\|} \right\rangle \right| \leq C [|P| + |\nabla E_P' |^n| + |C_{P,m}^{(k,n)}|]. \end{aligned}$$

where the last inequality holds by Lemma 5.9. \square

Lemma 6.5. *Let $|g|, \zeta, \gamma$ be sufficiently small. Furthermore, let $n \in \mathbb{N}$, $m \geq 2$ and $z \in \widehat{\Delta}_m$. Then*

$$\begin{aligned} & \left\| \left(\frac{1}{H_P^{W'} |_{m-1}^n - z} \right)^{1/2} \Delta \widehat{H}_n^{W'} |_{m-1}^{m-1} \left(\frac{1}{H_P^{W'} |_{m-1}^n - z} \right)^{1/2} \phi_P |_{m-1}^n \right\| \\ & \leq |g| C \gamma^{\frac{m-2}{2}} \left(1 + \sum_{i=1}^3 \left| \left\langle \Gamma_P^{(i)} |_{m-1}^n \phi_P |_{m-1}^n, \frac{1}{H_P^{W'} |_{m-1}^n - z} \Gamma_P^{(i)} |_{m-1}^n \phi_P |_{m-1}^n \right\rangle \right|^{\frac{1}{2}} \right) \end{aligned} \quad (130)$$

holds true, where $\Delta \widehat{H}_n^{W'} |_{m-1}^{m-1}$ is defined in (94).

Proof. Recall the expression for $\Delta \widehat{H}_n^{W'}|_m^{m-1}$ given in (95)-(97). With the usual estimates one can show that

$$\left\| \left(\frac{1}{H_P^{W'}|_{m-1}^n - z} \right)^{1/2} ((95) + (96)) \left(\frac{1}{H_P^{W'}|_{m-1}^n - z} \right)^{1/2} \phi_P|_{m-1}^n \right\| \leq |g|C\gamma^{\frac{m-1}{2}}. \quad (131)$$

Next, we control the first term in (97). First, observe that

$$\begin{aligned} & \left\| \left(\frac{1}{H_P^{W'}|_{m-1}^n - z} \right)^{1/2} (\widetilde{A}_{P,m}^{(n)} - A_{P,m-1}^{(n)}) \cdot \Gamma_P|_{m-1}^n \left(\frac{1}{H_P^{W'}|_{m-1}^n - z} \right)^{1/2} \phi_P|_{m-1}^n \right\|^2 \\ &= \frac{1}{|E_P'|_{m-1}^n - z|} \left\langle (\widetilde{A}_{P,m}^{(n)} - A_{P,m-1}^{(n)}) \cdot \Gamma_P|_{m-1}^n \phi_P|_{m-1}^n, \left| \frac{1}{H_P^{W'}|_{m-1}^n - z} \right| (\widetilde{A}_{P,m}^{(n)} - A_{P,m-1}^{(n)}) \cdot \Gamma_P|_{m-1}^n \phi_P|_{m-1}^n \right\rangle. \end{aligned} \quad (132)$$

Second, we recall that $\widetilde{A}_{P,m}^{(n)} - A_{P,m-1}^{(n)}$ contains boson creation operators restricted to the range $(\tau_m, \tau_{m-1}]$ in momentum space. Therefore,

$$\langle \phi_P|_{m-1}^n, \widetilde{A}_{P,m}^{(n)} - A_{P,m-1}^{(n)} \cdot \Gamma_P|_{m-1}^n \phi_P|_{m-1}^n \rangle = 0,$$

which implies

$$(132) \leq \frac{C}{\gamma^m} \left\| \left((\widetilde{A}_{P,m}^{(n)} - A_{P,m-1}^{(n)}) \cdot \Gamma_P|_{m-1}^n \phi_P|_{m-1}^n, \frac{1}{H_P^{W'}|_{m-1}^n - z} (\widetilde{A}_{P,m}^{(n)} - A_{P,m-1}^{(n)}) \cdot \Gamma_P|_{m-1}^n \phi_P|_{m-1}^n \right) \right\| \quad (133)$$

by using the spectral theorem and the gap estimate for $H_P^{W'}|_{m-1}^n \upharpoonright \mathcal{F}_m^n$. Note further that

$$(\widetilde{A}_{P,m}^{(n)} - A_{P,m-1}^{(n)}) \cdot \Gamma_P|_{m-1}^n \phi_P|_{m-1}^n = \int dk (\alpha_m(\nabla E_P'|_{m-1}^n) - \alpha_{m-1}(\nabla E_P'|_{m-1}^n)) b^*(k) k \cdot \Gamma_P|_{m-1}^n \phi_P|_{m-1}^n.$$

Using the pull-through formula we get

$$\frac{1}{H_P^{W'}|_{m-1}^n - z} b^*(k) = b^*(k) \frac{1}{H_P^{W'}|_{m-1}^n + \frac{1}{2}k^2 + k \cdot \Gamma_P|_{m-1}^n + |k| - \nabla E_P'|_{m-1}^n \cdot k - z}$$

so that we can rewrite the right-hand side of (133) as follows:

$$\begin{aligned} (133) &= \frac{C}{\gamma^m} \int dk [\alpha_m(\nabla E_P'|_{m-1}^n) - \alpha_{m-1}(\nabla E_P'|_{m-1}^n)]^2 \times \\ &\quad \times \left\langle k \cdot \Gamma_P|_{m-1}^n \phi_P|_{m-1}^n, \frac{1}{H_P^{W'}|_{m-1}^n + \frac{1}{2}k^2 + k \cdot \Gamma_P|_{m-1}^n + |k| - \nabla E_P'|_{m-1}^n \cdot k - z} k \cdot \Gamma_P|_{m-1}^n \phi_P|_{m-1}^n \right\rangle. \end{aligned} \quad (134)$$

In order to expand the resolvent in (134) in terms of $k \cdot \Gamma_P|_{m-1}^n$ we have to provide the bound

$$\left\| \left(\frac{1}{H_P^{W'}|_{m-1}^n + f_{P,m-1}(k) - z} \right)^{1/2} k \cdot \Gamma_P|_{m-1}^n \left(\frac{1}{H_P^{W'}|_{m-1}^n + f_{P,m-1}(k) - z} \right)^{1/2} \right\|_{\mathcal{F}_m^n} < 1 \quad (135)$$

for $\tau_m < |k| \leq \tau_{m-1}$ and $z \in \widehat{\Delta}_m$, where we have defined

$$f_{P,m-1}(k) := \frac{1}{2}k^2 + |k|(1 - \nabla E'_P|_{m-1}^n \cdot \widehat{k}).$$

Recall that

$$\Gamma_P|_{m-1}^n = P^f + A_{P,m-1}^{(n)} + B|_0^n + B^*|_0^n - \langle \Pi_P|_{m-1}^n \rangle_{\phi_P|_{m-1}^n}.$$

The necessary estimates are:

1. For $|g|$ sufficiently small, the lower bound

$$f_{P,m-1}(k) - |E'_P|_{m-1}^n - z| > |k| \left(1 - \nabla E'_P|_{m-1}^n \cdot \widehat{k} - \frac{1}{2}\zeta - g^2\gamma^{-1}C \right) > 0 \quad (136)$$

holds because z belongs to the shifted contour $\widehat{\Delta}_m$ so that

$$|E'_P|_{m-1}^n - z| \leq \frac{1}{2}\zeta\tau_m + g^2C\tau_{m-1}.$$

The inequality in (136) implies

$$\left\| \left(\frac{1}{H'_P|_{m-1}^n + f_{P,m-1}(k) - z} \right)^{1/2} \right\|_{\mathcal{F}|_{m-1}^n}^2 \leq \frac{1}{|k| \left(1 - \nabla E'_P|_{m-1}^n \cdot \widehat{k} - \frac{1}{2}\zeta - g^2\gamma^{-1}C \right)}.$$

2. By the unitarity of $W_{m-1}(\nabla E'_P|_{m-1}^n)$ and using $[B|_0^n, W_{m-1}(\nabla E'_P|_{m-1}^n)] = 0$ as well as the standard inequalities (22), we have

$$\left\| k \cdot B|_0^n \left(\frac{1}{H'_P|_{m-1}^n + f_{P,m-1}(k) - z} \right)^{1/2} \right\|_{\mathcal{F}|_{m-1}^n} \leq |g| |k| C \left\| H_{P,0}^{1/2} \left(\frac{1}{H'_P|_{m-1}^n + f_{P,m-1}(k) - z} \right)^{1/2} \right\|_{\mathcal{F}|_{m-1}^n}.$$

3. By definition of the transformation $W_{m-1}(\nabla E'_P|_{m-1}^n)$ and the transformation formulae (198),

$$W_{m-1}(\nabla E'_P|_{m-1}^n)(P - P^f)W_{m-1}(\nabla E'_P|_{m-1}^n)^* = P - P^f - A_{P,m-1}^{(n)} - C_{P,m-1}^{(k,n)}$$

holds on $D(H_{P,0})$. Hence, we have the bound

$$\begin{aligned} & \left\| k \cdot (P^f + A_{P,m-1}^{(n)}) \left(\frac{1}{H'_P|_{m-1}^n + f_{P,m-1}(k) - z} \right)^{1/2} \right\|_{\mathcal{F}|_{m-1}^n} \\ & \leq |k| \left\| (P - P^f) \left(\frac{1}{H'_P|_{m-1}^n + f_{P,m-1}(k) - z} \right)^{1/2} \right\|_{\mathcal{F}|_{m-1}^n} \\ & \quad + |k|(|P| + g^2C) \left\| \left(\frac{1}{H'_P|_{m-1}^n + f_{P,m-1}(k) - z} \right)^{1/2} \right\|_{\mathcal{F}|_{m-1}^n} \\ & \leq |k| \sqrt{2} \left\| H_{P,0}^{1/2} \left(\frac{1}{H'_P|_{m-1}^n + f_{P,m-1}(k) - z} \right)^{1/2} \right\|_{\mathcal{F}|_{m-1}^n} \\ & \quad + |k|(|P| + g^2C) \left\| \left(\frac{1}{H'_P|_{m-1}^n + f_{P,m-1}(k) - z} \right)^{1/2} \right\|_{\mathcal{F}|_{m-1}^n}. \end{aligned}$$

4. Using the a priori estimate (23) in Lemma 3.2 one derives

$$\begin{aligned} & \left\| H_{P,0}^{1/2} \left(\frac{1}{H_P'^n|_{m-1} + f_{P,m-1}(k) - z} \right)^{1/2} \right\|_{\mathcal{F}|_{m-1}^n} \\ & \leq \frac{1}{\sqrt{1-|g|c_a}} \left(\left\| (H_P'^n|_{m-1})^{1/2} \left(\frac{1}{H_P'^n|_{m-1} + f_{P,m-1}(k) - z} \right)^{1/2} \right\|_{\mathcal{F}|_{m-1}^n}^2 \right. \\ & \quad \left. + |g|c_b \left\| \left(\frac{1}{H_P'^n|_{m-1} + f_{P,m-1}(k) - z} \right)^{1/2} \right\|_{\mathcal{F}|_{m-1}^n}^2 \right)^{1/2}. \end{aligned}$$

Collecting these estimates, we find:

$$\left\| \left(\frac{1}{H_P^{W'}|_{m-1} + f_{P,m-1}(k) - z} \right)^{1/2} k \cdot \Gamma_P|_{m-1} \left(\frac{1}{H_P^{W'}|_{m-1} + f_{P,m-1}(k) - z} \right)^{1/2} \right\|_{\mathcal{F}|_{m-1}^n} \quad (137)$$

$$\begin{aligned} & \leq |k| \left\| \left(\frac{1}{H_P^{W'}|_{m-1} + f_{P,m-1}(k) - z} \right)^{1/2} \right\|_{\mathcal{F}|_{m-1}^n} \times \quad (138) \\ & \times \left[\frac{\sqrt{2} + |g|C}{\sqrt{1-|g|c_a}} \left(\left\| (H_P'^n|_{m-1})^{1/2} \left(\frac{1}{H_P'^n|_{m-1} + f_{P,m-1}(k) - z} \right)^{1/2} \right\|_{\mathcal{F}|_{m-1}^n}^2 \right. \right. \\ & \quad \left. \left. + |g|c_b \left\| \left(\frac{1}{H_P'^n|_{m-1} + f_{P,m-1}(k) - z} \right)^{1/2} \right\|_{\mathcal{F}|_{m-1}^n}^2 \right)^{1/2} + (|P| + g^2C) \left\| \left(\frac{1}{H_P|_{m-1} + f_{P,m-1}(k) - z} \right)^{1/2} \right\|_{\mathcal{F}|_{m-1}^n} \right]. \end{aligned}$$

Note that

$$\left\| (H_P'^n|_{m-1})^{1/2} \left(\frac{1}{H_P'^n|_{m-1} + f_{P,m-1}(k) - z} \right)^{1/2} \right\|_{\mathcal{F}|_{m-1}^n} \leq \left(1 + \frac{|E_P'^n|_{m-1}|}{f_{P,m-1}(k) - |E_P'^n|_{m-1} - z} \right)^{1/2}.$$

Finally we obtain

$$\begin{aligned} (137) & \leq \frac{1}{\left(1 - \nabla E_P'^n|_{m-1} \cdot \widehat{k} - \frac{1}{2}\zeta - g^2\gamma^{-1}C \right)} \times \\ & \times \left[\frac{\sqrt{2} + |g|C}{\sqrt{1-|g|c_a}} \left(|E_P'^n|_{m-1}| + \tau_{m-1} \left(1 - \nabla E_P'^n|_{m-1} \cdot \widehat{k} - \frac{1}{2}\zeta - Cg^2\gamma^{-1} \right) + gc_b \right)^{1/2} + (|P| + g^2C) \right] \end{aligned}$$

so that

$$\lim_{|g|, \gamma, \zeta \rightarrow 0} \sup (137) \leq \frac{2P_{\max}}{1 - P_{\max}} < \frac{2}{3}$$

for $P_{\max} < \frac{1}{4}$. By continuity, inequality (135) holds for g, ζ, γ in a neighborhood of zero.

Going back to equation (134) we can proceed with the expansion (in $k \cdot \Gamma_P|_{m-1}^n$) of the resolvent:

$$(133) \leq C g^2 \gamma^{m-2} \sup_{\tau_m \leq |k| \leq \tau_{m-1}} \sum_{i,l=1}^3 \left\langle \left[\left(\frac{1}{H_P^{W'}|_{m-1}^n + f_{P,m-1}(k) - z} \right)^{1/2} \right]^* \Gamma_P^{(i)}|_{m-1}^n \phi_P|_{m-1}^n \right\rangle, \quad (139)$$

$$\begin{aligned} & \sum_{j=0}^{\infty} \left[\left(\frac{1}{H_P^{W'}|_{m-1}^n + f_{P,m-1}(k) - z} \right)^{1/2} k \cdot \Gamma_P|_{m-1}^n \left(\frac{1}{H_P^{W'}|_{m-1}^n + f_{P,m-1}(k) - z} \right)^{1/2} \right]^j \times \\ & \times \left(\frac{1}{H_P^{W'}|_{m-1}^n + f_{P,m-1}(k) - z} \right)^{1/2} \Gamma_P^{(l)}|_{m-1}^n \phi_P|_{m-1}^n \Big\rangle \\ & \leq C g^2 \gamma^{m-2} \sum_{i=1}^3 \sup_{\tau_m \leq |k| \leq \tau_{m-1}} \left\| \left(\frac{1}{H_P^{W'}|_{m-1}^n + f_{P,m-1}(k) - z} \right)^{1/2} \Gamma_P^{(i)}|_{m-1}^n \phi_P|_{m-1}^n \right\|^2. \end{aligned} \quad (140)$$

Since $f_{P,m-1}(k) \geq 0$ and because of the property $\langle \phi_P|_{m-1}^n, \Gamma_P|_{m-1}^n \phi_P|_{m-1}^n \rangle = 0$ it follows that

$$\left\| \left(\frac{1}{H_P^{W'}|_{m-1}^n + f_{P,m-1}(k) - z} \right)^{1/2} \Gamma_P^{(i)}|_{m-1}^n \phi_P|_{m-1}^n \right\|^2 \leq C \left\| \left\langle \Gamma_P^{(i)}|_{m-1}^n \phi_P|_{m-1}^n, \frac{1}{H_P^{W'}|_{m-1}^n - z} \Gamma_P^{(i)}|_{m-1}^n \phi_P|_{m-1}^n \right\rangle \right\|.$$

Combining the estimates in (140) and (131) yields the claim of the lemma. \square

Lemma 6.6. For all $n, m \in \mathbb{N}$ and $Q, Q' \in \mathbb{R}^3$ with $|Q|, |Q'| \leq 1$ the estimate

$$\| [W_m(Q) - W_m(Q')] \Psi_P'|_m^n \| \leq |g|C|Q - Q'| \ln \tau_m$$

holds.

Proof. The Bogolyubov transformations W_m defined in (14) can be explicitly written as

$$W_m(Q) = \exp \left(\int dk \alpha_m(Q, k) (b(k) - b^*(k)) \right),$$

so that

$$\| [W_m(Q) - W_m(Q')] \Psi_P'|_m^n \| \leq \left\| \int dk [\alpha_m(Q, k) - \alpha_m(Q', k)] (b(k) - b^*(k)) \Psi_P'|_m^n \right\| \quad (141)$$

In order to estimate this term we employ:

1. The identity (12) in [Frö73, Equation (1.26)] that relies on the bound $E'_{P-k}|_m^n - E'_P|_m^n \geq -C_{\nabla E}|k|$, $|P| \leq P_{\max}$, from Corollary 5.4(iii).
2. By definition of α_m it holds

$$\int dk |\alpha_m(Q, k) - \alpha_m(Q', k)| \frac{1}{|k|^{3/2}} \leq |g|C|Q - Q'| |\ln \kappa - \ln \tau_m|.$$

3. $\| \Psi_P'|_m^n \| \leq 1$

With these estimates, the claim is proven. \square

Lemma 6.7. *Let $|g|$ be sufficiently small. For $n, m \in \mathbb{N}$ the following estimate holds:*

$$\begin{aligned} & |\nabla E'_{P|m} - \nabla E'_{P|m-1}| \\ & \leq g^2 C \tau_{m-1}^{1/2} + C \sup_{z \in \widehat{\Delta}_m} \left\| \left(\frac{1}{H_P^{W'}|_{m-1} - z} \right)^{1/2} \Delta \widehat{H}_n^{W'}|_m^{m-1} \left(\frac{1}{H_P^{W'}|_{m-1} - z} \right)^{1/2} \phi_{P|m-1} \right\| + C \frac{\|\phi_{P|m-1} - \widetilde{\phi}_{P|m}\|}{\|\phi_{P|m-1}\|^2 \|\widetilde{\phi}_{P|m}\|^2}. \end{aligned}$$

Proof. Let $n, m \in \mathbb{N}$. Using Lemma 5.9 we have

$$\nabla E'_{P|m} - \nabla E'_{P|m-1} = \langle P^f + B|_0^n + B^*|_0^n \rangle_{\Psi_P'|_m} - \langle P^f + B|_0^n + B^*|_0^n \rangle_{\Psi_P'|_{m-1}}$$

which by unitarity of the transformation $W_{m-1}(\nabla E'_{P|m-1})$ and $W_m(\nabla E'_{P|m})$ can be rewritten as

$$\nabla E'_{P|m} - \nabla E'_{P|m-1} = \langle \Pi_P|_{m-1}^n \rangle_{\phi_P|_{m-1}^n} - \langle \widetilde{\Pi}_P|_m^n \rangle_{\widetilde{\phi}_P|_m^n} + \widetilde{C}_{P,m}^{(k,n)} - C_{P,m-1}^{(k,n)}.$$

We have already noted that $|\widetilde{C}_{P,m}^{(k,n)} - C_{P,m-1}^{(k,n)}| \leq g^2 C \tau_{m-1}$. Moreover, we observe

$$\begin{aligned} & \left| \langle \Pi_P|_{m-1}^n \rangle_{\phi_P|_{m-1}^n} - \langle \widetilde{\Pi}_P|_m^n \rangle_{\widetilde{\phi}_P|_m^n} \right| = \left| \frac{\langle \phi_P|_{m-1}^n, \Pi_P|_{m-1}^n \phi_P|_{m-1}^n \rangle}{\|\phi_P|_{m-1}^n\|^2} - \frac{\langle \widetilde{\phi}_P|_m^n, \widetilde{\Pi}_P|_m^n \widetilde{\phi}_P|_m^n \rangle}{\|\widetilde{\phi}_P|_m^n\|^2} \right| \\ & \leq \|\phi_P|_{m-1}^n\|^{-2} \left| \langle \phi_P|_{m-1}^n, \Pi_P|_{m-1}^n \phi_P|_{m-1}^n \rangle - \langle \widetilde{\phi}_P|_m^n, \widetilde{\Pi}_P|_m^n \widetilde{\phi}_P|_m^n \rangle \right| + \end{aligned} \quad (142)$$

$$+ \left| \langle \widetilde{\phi}_P|_m^n, \widetilde{\Pi}_P|_m^n \widetilde{\phi}_P|_m^n \rangle \right| \left| \|\phi_P|_{m-1}^n\|^{-2} - \|\widetilde{\phi}_P|_m^n\|^{-2} \right|. \quad (143)$$

We know that the norms $\|\phi_P|_{m-1}^n\|$ and $\|\widetilde{\phi}_P|_m^n\|$ are by construction smaller than one and non-zero. Using Lemma 6.4 we find

$$(143) \leq C \frac{\|\phi_P|_{m-1}^n - \widetilde{\phi}_P|_m^n\|}{\|\phi_P|_{m-1}^n\|^2 \|\widetilde{\phi}_P|_m^n\|^2}.$$

In order to bound the term (142) we use

$$\|\phi_P|_{m-1}^n\|^2 (142) = \left| \langle (\phi_P|_{m-1}^n - \widetilde{\phi}_P|_m^n), \Pi_P|_{m-1}^n \phi_P|_{m-1}^n \rangle + \right. \quad (144)$$

$$+ \langle \widetilde{\phi}_P|_m^n, [\Pi_P|_{m-1}^n - \widetilde{\Pi}_P|_m^n] \phi_P|_{m-1}^n \rangle + \quad (145)$$

$$\left. + \langle \widetilde{\phi}_P|_m^n, \widetilde{\Pi}_P|_m^n (\phi_P|_{m-1}^n - \widetilde{\phi}_P|_m^n) \rangle \right|. \quad (146)$$

The term (145) is bounded by

$$|(145)| \leq \left| \langle \widetilde{\phi}_P|_m^n, [\widetilde{A}_{P,m}^{(n)} - A_{P,m-1}^{(n)}] \phi_P|_{m-1}^n \rangle \right| \leq |g| C \tau_{m-1}^{1/2}$$

because by the standard inequalities (22)

$$\begin{aligned} & \left\| \int dk \, k [\alpha_m(\nabla E'_P|_{m-1}^n, k) - \alpha_{m-1}(\nabla E'_P|_{m-1}^n, k)] b(k) \phi_P|_{m-1}^n \right\| \\ & \leq C \left(\int dk \, \left| \frac{k [\alpha_m(\nabla E'_P|_{m-1}^n, k) - \alpha_{m-1}(\nabla E'_P|_{m-1}^n, k)]^2}{|k|^{1/2}} \right|^{1/2} \right) \left\| (H_P^{f|_{m-1}}) \left(\frac{1}{H_P^{W'|_{m-1}} - i} \right)^{1/2} \phi_P|_{m-1}^n \right\| \leq |g| C \tau_{m-1}^{1/2}. \end{aligned}$$

Terms (144) and (146) can be treated in the same way, and we only demonstrate the bound on the former. Using analytic perturbation theory we get

$$\left| \left\langle (\phi_P|_{m-1}^n - \tilde{\phi}_P|_m^n), \Pi_P|_{m-1}^n \phi_P|_{m-1}^n \right\rangle \right| \quad (147)$$

$$\begin{aligned} & \leq C \tau_m \sup_{z \in \tilde{\Delta}_m} \sum_{j=1}^{\infty} \left| \left\langle \left[\left(\frac{1}{H_P^{W'|_{m-1}} - z} \right)^{1/2} \Delta \widehat{H}_n^{W'|_{m-1}} \left(\frac{1}{H_P^{W'|_{m-1}} - z} \right)^{1/2} \right]^j \phi_P|_{m-1}^n, \right. \right. \\ & \quad \left. \left. \left[\left(\frac{1}{H_P^{W'|_{m-1}} - z} \right)^{1/2} \right]^* \Pi_P|_{m-1}^n \left(\frac{1}{H_P^{W'|_{m-1}} - z} \right)^{1/2} \phi_P|_{m-1}^n \right\rangle \right| \\ & \leq C \tau_m \sup_{z \in \tilde{\Delta}_m} \left\| \left(\frac{1}{H_P^{W'|_{m-1}} - z} \right)^{1/2} \Delta \widehat{H}_n^{W'|_{m-1}} \left(\frac{1}{H_P^{W'|_{m-1}} - z} \right)^{1/2} \phi_P|_{m-1}^n \right\| \times \\ & \quad \times \left\| \left[\left(\frac{1}{H_P^{W'|_{m-1}} - z} \right)^{1/2} \right]^* \Pi_P|_{m-1}^n \left(\frac{1}{H_P^{W'|_{m-1}} - z} \right)^{1/2} \right\|_{\mathcal{F}_m^n}. \end{aligned} \quad (148)$$

The term in (148) can be controlled similarly to (127) in the ultraviolet regime so that we finally have

$$\left\| \left[\left(\frac{1}{H_P^{W'|_{m-1}} - z} \right)^{1/2} \right]^* \Pi_P|_{m-1}^n \left(\frac{1}{H_P^{W'|_{m-1}} - z} \right)^{1/2} \right\|_{\mathcal{F}_m^n} \leq C \tau_m^{-1}. \quad (149)$$

Combining these results, we obtain the estimate

$$\left| \left\langle (\phi_P|_{m-1}^n - \tilde{\phi}_P|_m^n), \Pi_P|_{m-1}^n \phi_P|_{m-1}^n \right\rangle \right| \leq C \sup_{z \in \tilde{\Delta}_m} \left\| \left(\frac{1}{H_P^{W'|_{m-1}} - z} \right)^{1/2} \Delta \widehat{H}_n^{W'|_{m-1}} \left(\frac{1}{H_P^{W'|_{m-1}} - z} \right)^{1/2} \phi_P|_{m-1}^n \right\|,$$

which concludes the proof. \square

7 Ground States of the Transformed Hamiltonians $H_P^{W'}|_{\infty}^{\infty}$

In this section, we finally remove both the UV and the IR cut-off (σ_n and τ_m , respectively). In our study of the removal of the IR cut-off in Section 6 we have proven that

$$\|\phi_P|_m^n - \phi_P|_{m-1}^n\| \leq (m+1) \gamma^{\frac{m}{4}}$$

holds for any $n \in \mathbb{N}$. We shall now provide the analogous bound

$$\|\phi_P|_m^n - \phi_P|_m^{n-1}\| \leq C m K^{3m+1} |\ln \gamma|^{m+1} \left(\frac{n}{\beta^n \gamma^m} \right)^{1/2} \quad (150)$$

as the UV cut-off is shifted from σ_{n-1} to σ_n . The constant $K \geq 1$ will be introduced in Theorem 7.5. The latter bound, derived in Corollary 7.6, holds for any IR cut-off τ_m and uses a particular scaling $\mathbb{N} \ni n := n(m) > \alpha m$ for

$$\alpha := \frac{-\ln |\gamma|}{\ln \beta} \geq 1. \quad (151)$$

These two estimates will enable us to prove the second main result Theorem 2.2 at the end of this section.

Remark 7.1. *In this section we implicitly assume the constraints $|P| < P_{\max}$ and $1 < \kappa < 2$. Furthermore, g, β , and γ are such that all the results of Sections 4, 5, and 6 hold true.*

In order to control the norm difference $\|\phi_{P|m}^n - \phi_{P|m-1}^n\|$ we notice that for $m \geq 1$ the vectors $\phi_{P|m}^n$ can be rewritten in the following way

$$\phi_{P|m}^n = W_m(\nabla E'_{P|m})^n Q'_{P|m} W_m^{m-1}(\nabla E'_{P|m-1})^* \cdots Q'_{P|2} W_2^1(\nabla E'_{P|1})^* Q'_{P|1} W_1^0(\nabla E'_{P|0})^* \frac{\Psi'_{P|0}^n}{\|\Psi'_{P|0}\|^n},$$

where $Q'_{P|m}$ is defined in (76) and

$$W_m^{m'}(Q)^* := W_m(Q)^* W_{m'}(Q), \quad W_1^0(Q)^* = W_1(Q).$$

The following definition will be convenient.

Definition 7.2. *For $n \in \mathbb{N}$ and $m \geq 1$, we define*

$$\eta_{P|m}^n := W_m(\nabla E'_{P|m})^n \phi_{P|m}^n, \quad (152)$$

and $\eta_{P|0}^n := \phi_{P|0}^n = \Psi_{P|0}^n / \|\Psi_{P|0}\|^n$ in the case $m = 0$.

Note that by construction we have the identity

$$\eta_{P|m+1}^n = Q'_{P|m+1} W_{m+1}^m(\nabla E'_{P|m})^* \eta_{P|m}^n, \quad (153)$$

and moreover, since the transformation W_m is unitary and due to Theorem 6.3, the bounds

$$1 \geq \|\phi_{P|m}^n\| = \|\eta_{P|m}^n\| \geq \frac{1}{2} \quad (154)$$

hold true for all $m, n \in \mathbb{N}$. First, we prove two a priori lemmas that can be combined to yield Theorem 7.5.

Lemma 7.3. *For any $m \in \mathbb{N}$, let $\mathbb{N} \ni n > \alpha m \geq 1$. There exists a constant K_1 such that for $|g|$ sufficiently small the following estimates hold true:*

$$\|\eta_{P|m+1}^n - \eta_{P|m+1}^{n-1}\| \leq \|\eta_{P|m}^n - \eta_{P|m}^{n-1}\| + K_1 \left[\left(\frac{n}{\beta^n \gamma^{m+1}} \right)^{1/2} + |\nabla E'_{P|m} - \nabla E'_{P|m-1}| \right]. \quad (155)$$

Proof. By using (152) and (153) we get the bound

$$\|\eta_P|_{m+1}^n - \eta_P|_{m+1}^{n-1}\| \leq \left\| \left(Q_P'|_{m+1}^n - Q_P'|_{m+1}^{n-1} \right) W_{m+1}^m (\nabla E_P'|_m^n)^* \eta_P|_m^n \right\| \quad (156)$$

$$+ \left\| Q_P'|_{m+1}^{n-1} \left(W_{m+1}^m (\nabla E_P'|_m^n)^* - W_{m+1}^m (\nabla E_P'|_m^{n-1})^* \right) \eta_P|_m^n \right\| \quad (157)$$

$$+ \left\| Q_P'|_{m+1}^{n-1} W_{m+1}^m (\nabla E_P'|_m^{n-1})^* \left(\eta_P|_m^n - \eta_P|_m^{n-1} \right) \right\|. \quad (158)$$

Furthermore, the expansion

$$\begin{aligned} Q_P'|_{m+1}^n - Q_P'|_{m+1}^{n-1} = & -\frac{1}{2\pi i} \oint_{\Delta_{m+1}} dz \left\{ \left(\frac{1}{H_P'|_{m+1}^{n-1} - z} \right)^{1/2} \times \right. \\ & \times \sum_{j=1}^{\infty} \left[\left(\frac{1}{H_P'|_{m+1}^{n-1} - z} \right)^{1/2} \Delta H'|_{n-1} \left(\frac{1}{H_P'|_{m+1}^{n-1} - z} \right)^{1/2} \right]^j \times \\ & \left. \times \left(\frac{1}{H_P'|_{m+1}^{n-1} - z} \right)^{1/2} \right\}, \end{aligned} \quad (159)$$

can be controlled by noting that

$$\left\| \left(\frac{1}{H_P'|_{m+1}^{n-1} - z} \right)^{1/2} \right\|_{\mathcal{F}|_{m+1}^n}^2 \leq \frac{2}{\zeta \tau_{m+1}} \quad (160)$$

(see Lemma 5.5), which yields

$$\sup_{z \in \Delta_{m+1}} \left\| \left(\frac{1}{H_P'|_{m+1}^{n-1} - z} \right)^{1/2} \Delta H'|_{n-1} \left(\frac{1}{H_P'|_{m+1}^{n-1} - z} \right)^{1/2} \right\|_{\mathcal{F}|_{m+1}^n} \leq C|g| \left(\frac{n}{\beta^n \zeta \tau_{m+1}} \right)^{1/2} \quad (161)$$

by a similar computation as for (50). Now, by the choice $n > \alpha m$ and $|g|$ sufficiently small, the right-hand side in (161) is strictly smaller than 1. Hence, we get

$$\|Q_P'|_{m+1}^n - Q_P'|_{m+1}^{n-1}\| \leq C|g| \left(\frac{n}{\beta^n \zeta \tau_{m+1}} \right)^{1/2}.$$

Moreover, under the constraint in (68) we get the bound

$$(157) \leq C|g| |\ln \gamma| |\nabla E_P'|_m^n - \nabla E_P'|_m^{n-1}| \leq C |\nabla E_P'|_m^n - \nabla E_P'|_m^{n-1}|$$

by a similar procedure as used in the proof of Lemma 6.6. The remaining term (158) can be estimated using the unitarity of W_m . This concludes the proof. \square

Lemma 7.4. *For any $m \in \mathbb{N}$, let $\mathbb{N} \ni n > \alpha m \geq 1$. There exists a constant K_2 such that for $|g|$ sufficiently small the following estimate holds true:*

$$|\nabla E_P'|_m^n - \nabla E_P'|_m^{n-1}| \leq K_2 \left[\left(\frac{n}{\beta^n \gamma^m} \right)^{1/2} + \|\eta_P|_m^n - \eta_P|_m^{n-1}\| + |\nabla E_P'|_{m-1}^n - \nabla E_P'|_{m-1}^{n-1}| \right]. \quad (162)$$

Proof. Let us start with the equality

$$|\nabla E'_{P|_m}|^n - \nabla E'_{P|_m}|^{n-1}| = \left| \left\langle P^f + B|_0^n + B^*|_0^n \right\rangle_{\Psi'_{P|_m}|^n} - \left\langle P^f + B|_0^{n-1} + B^*|_0^{n-1} \right\rangle_{\Psi'_{P|_m}|^{n-1}} \right|. \quad (163)$$

As $\Psi'_{P|_m}|^n$ and $\eta_{P|_m}|^n$ belong to the same ray in \mathcal{H}_P , we obtain

$$(163) = \left| \left\langle P^f + B|_0^n + B^*|_0^n \right\rangle_{\eta_{P|_m}|^n} - \left\langle P^f + B|_0^{n-1} + B^*|_0^{n-1} \right\rangle_{\eta_{P|_m}|^{n-1}} \right|.$$

In order to shorten the formulae we define

$$V_n := P^f + B|_0^n + B^*|_0^n$$

so that

$$(163) \leq \frac{1}{\|\eta_{P|_m}|^{n-1}\|^2} \left| \langle \eta_{P|_m}|^n, V_n \eta_{P|_m}|^n \rangle - \langle \eta_{P|_m}|^{n-1}, V_{n-1} \eta_{P|_m}|^{n-1} \rangle \right| \quad (164)$$

$$+ \left| \frac{1}{\|\eta_{P|_m}|^n\|^2} - \frac{1}{\|\eta_{P|_m}|^{n-1}\|^2} \right| \left| \langle \eta_{P|_m}|^n, V_n \eta_{P|_m}|^n \rangle \right|. \quad (165)$$

Furthermore, by the definitions in (82), (83) and (152) we have

$$\left| \langle \eta_{P|_m}|^n, V_n \eta_{P|_m}|^n \rangle \right| = \left| \langle \phi_{P|_m}|^n, \Pi_{P|_m}|^n \phi_{P|_m}|^n \rangle + C_{P,m}^{(k,n)} \|\phi_{P|_m}|^n\|^2 \right| \leq C, \quad (166)$$

where we used Lemma 6.4. Hence, by (154) we get the estimate

$$(165) \leq C \frac{\|\eta_{P|_m}|^n - \eta_{P|_m}|^{n-1}\|}{\|\eta_{P|_m}|^n\|^2 \|\eta_{P|_m}|^{n-1}\|^2} \leq C \|\eta_{P|_m}|^n - \eta_{P|_m}|^{n-1}\|. \quad (167)$$

Next, we proceed with

$$(164) \leq C \left[\left| \langle (\eta_{P|_m}|^n - \eta_{P|_m}|^{n-1}), V_n \eta_{P|_m}|^n \rangle \right| \right. \quad (168)$$

$$+ \left| \langle \eta_{P|_m}|^{n-1}, (V_n - V_{n-1}) \eta_{P|_m}|^n \rangle \right| \quad (169)$$

$$\left. + \left| \langle \eta_{P|_m}|^{n-1}, V_{n-1} (\eta_{P|_m}|^n - \eta_{P|_m}|^{n-1}) \rangle \right| \right]. \quad (170)$$

First, we observe that

$$(169) \leq C \left| \langle \eta_{P|_m}|^{n-1}, (B|_{n-1}^n + B^*|_{n-1}^n) \eta_{P|_m}|^n \rangle \right| \leq C |E'_{P|_m}|^n - i|^{1/2} \left| \left\langle \eta_{P|_m}|^{n-1}, B|_{n-1}^n \left(\frac{1}{H'_{P|_m}|^n - i} \right)^{1/2} \eta_{P|_m}|^n \right\rangle \right|$$

holds. Invoking the standard inequalities in (39) and the boundedness of

$$\left\| H_{P,0}^{1/2} \left(\frac{1}{H'_{P|_m}|^n - i} \right)^{1/2} \right\| \leq C, \quad (171)$$

which holds by Lemma 3.2, one has

$$\left\| B_{|n-1}^n \left(\frac{1}{H_{P|n-1}^n - i} \right)^{1/2} \right\|_{\mathcal{F}_{|n-1}^n} \leq C |g| \left(\frac{1}{\beta^n} \right)^{1/2}.$$

Hence, since the ground state energies are bounded from above and below by Corollary 5.4,

$$(169) \leq C \left(\frac{1}{\beta^n} \right)^{1/2} \quad (172)$$

holds true. Terms (168) and (170) can be treated similarly. By recalling the identity in (153) we can write

$$(168) = \left| \left\langle (Q_{P|n-1}^n W_{|n-1}^{m-1} (\nabla E_{P|n-1}^n)^* \eta_{P|n-1}^n - Q_{P|n-1}^{n-1} W_{|n-1}^{m-1} (\nabla E_{P|n-1}^{n-1})^* \eta_{P|n-1}^{n-1}), V_n \eta_{P|n-1}^n \right\rangle \right|$$

$$\leq \left| \left\langle (Q_{P|n-1}^n - Q_{P|n-1}^{n-1}) W_{|n-1}^{m-1} (\nabla E_{P|n-1}^n)^* \eta_{P|n-1}^n, V_n \eta_{P|n-1}^n \right\rangle \right| \quad (173)$$

$$+ \left| \left\langle Q_{P|n-1}^{n-1} (W_{|n-1}^{m-1} (\nabla E_{P|n-1}^n)^* - W_{|n-1}^{m-1} (\nabla E_{P|n-1}^{n-1})^*) \eta_{P|n-1}^n, V_n \eta_{P|n-1}^n \right\rangle \right| \quad (174)$$

$$+ \left| \left\langle Q_{P|n-1}^{n-1} W_{|n-1}^{m-1} (\nabla E_{P|n-1}^{n-1})^* (\eta_{P|n-1}^n - \eta_{P|n-1}^{n-1}), V_n \eta_{P|n-1}^n \right\rangle \right|. \quad (175)$$

Observe that

$$\begin{aligned} & \left\langle Q_{P|n-1}^{n-1} W_{|n-1}^{m-1} (\nabla E_{P|n-1}^{n-1})^* (\eta_{P|n-1}^n - \eta_{P|n-1}^{n-1}), V_n \eta_{P|n-1}^n \right\rangle \\ &= \left\langle W_{|n-1}^{m-1} (\nabla E_{P|n-1}^{n-1})^* (\eta_{P|n-1}^n - \eta_{P|n-1}^{n-1}), Q_{P|n-1}^{n-1} V_n \eta_{P|n-1}^n \right\rangle \\ &= \frac{1}{\|\eta_{P|n-1}^{n-1}\|^2} \left\langle W_{|n-1}^{m-1} (\nabla E_{P|n-1}^{n-1})^* (\eta_{P|n-1}^n - \eta_{P|n-1}^{n-1}), \eta_{P|n-1}^{n-1} \right\rangle \left\langle \eta_{P|n-1}^{n-1}, V_n \eta_{P|n-1}^n \right\rangle. \end{aligned}$$

With

$$\begin{aligned} \left| \left\langle \eta_{P|n-1}^{n-1}, V_n \eta_{P|n-1}^n \right\rangle \right| &\leq C |E_{P|n-1}^n - i|^{1/2} \left| \left\langle \eta_{P|n-1}^{n-1}, H_{P,0}^{1/2} \left(\frac{1}{H_{P|n-1}^n - i} \right)^{1/2} \eta_{P|n-1}^n \right\rangle \right| \\ &+ C |E_{P|n-1}^{n-1} - i|^{1/2} \left| \left\langle \eta_{P|n-1}^{n-1}, \left(\frac{1}{H_{P|n-1}^{n-1} - i} \right)^{1/2} H_{P,0}^{1/2} \eta_{P|n-1}^n \right\rangle \right| \end{aligned}$$

and (171), we obtain the first estimate

$$(175) \leq C \|\eta_{P|n-1}^n - \eta_{P|n-1}^{n-1}\| \left| \left\langle \eta_{P|n-1}^{n-1}, V_n \eta_{P|n-1}^n \right\rangle \right| \leq C \|\eta_{P|n-1}^n - \eta_{P|n-1}^{n-1}\|.$$

Furthermore, (174) can be bounded by

$$\begin{aligned} (174) &\leq C \left\| (W_{|n-1}^{m-1} (\nabla E_{P|n-1}^n)^* - W_{|n-1}^{m-1} (\nabla E_{P|n-1}^{n-1})^*) \eta_{P|n-1}^n \right\| \left| \left\langle \eta_{P|n-1}^{n-1}, V_n \eta_{P|n-1}^n \right\rangle \right| \\ &\leq C |g| \ln |g| \left| \nabla E_{P|n-1}^n - \nabla E_{P|n-1}^{n-1} \right| \leq C \left| \nabla E_{P|n-1}^n - \nabla E_{P|n-1}^{n-1} \right| \end{aligned}$$

where the constraints (68) has been used again. Finally, using the resolvent expansion in (159) we get

$$(173) \leq C \tau_m^{\frac{1}{2}} \left(\frac{n}{\beta^n \gamma^m} \right)^{1/2} |E_{P|n-1}^n - i|^{1/2} \sup_{z \in \Delta_m} \left\| \left[\left(\frac{1}{H_{P|n-1}^{n-1} - z} \right)^{1/2} \right]^* V_n \left(\frac{1}{H_{P|n-1}^n - i} \right)^{1/2} \eta_{P|n-1}^n \right\|$$

and the standard inequalities in (22) and Lemma 3.2 yield

$$(173) \leq C \left(\frac{n}{\beta^n \gamma^m} \right)^{1/2}.$$

Carrying out the same argument for term (170) one obtains

$$(168) + (170) \leq C \left[\left(\frac{n}{\beta^n \gamma^m} \right)^{1/2} + \|\eta_P|_m^n - \eta_P|_m^{n-1}\| + |\nabla E'_P|_m^n - \nabla E'_P|_m^{n-1}| \right]$$

which, together with estimate (172), proves the claim. \square

Theorem 7.5. *There exist constants $K \geq \max(K_1, K_2, 5)$, $g_* > 0$ and $\frac{1}{2} > \gamma_* > 0$ such that for $|g| \leq g_*$ and $\gamma \leq \gamma_*$ the following estimates hold true for all finite $n \in \mathbb{N}$ and $\mathbb{N} \ni m < n/\alpha$:*

$$(i) \quad |\nabla E'_P|_m^n - \nabla E'_P|_m^{n-1}| \leq K^{3m+1} \left(\frac{n}{\beta^n \gamma^m} \right)^{1/2}.$$

$$(ii) \quad \|\eta_P|_m^n - \eta_P|_m^{n-1}\| \leq K^{3m+1} \left(\frac{n}{\beta^n \gamma^m} \right)^{1/2}.$$

Proof. Let $n \in \mathbb{N}$ and fix $K \geq \max(K_1, K_2, 5)$. We prove the claim by induction in m for $m < n/\alpha$. Statements (i)-(ii) for m will be referred to as assumptions A(i)-A(ii) while the same statements for $m+1$ are claims C(i)-C(ii). We recall that $\eta_P|_0^n \equiv \phi_P|_0^n \equiv \Psi'_P|_0^n / \|\Psi'_P|_0^n\|$ so that C(i) and C(ii) for $m=0$ are consequence of (58) and (55) for $|g|$ sufficiently small. The induction step $m \Rightarrow (m+1)$ for $(m+1) < \frac{n}{\alpha}$ is a straightforward consequence of inequalities (162) and (155): For C(i) we estimate

$$\begin{aligned} |\nabla E'_P|_{m+1}^n - \nabla E'_P|_{m+1}^{n-1}| &\leq K_2 \left[\left(\frac{n}{\beta^n \gamma^{m+1}} \right)^{1/2} + \|\eta_P|_{m+1}^n - \eta_P|_{m+1}^{n-1}\| + |\nabla E'_P|_m^n - \nabla E'_P|_m^{n-1}| \right] \\ &\leq K_2 \left[\left(\frac{n}{\beta^n \gamma^{m+1}} \right)^{1/2} + |\nabla E'_P|_m^n - \nabla E'_P|_m^{n-1}| \right. \\ &\quad \left. + \|\eta_P|_m^n - \eta_P|_m^{n-1}\| + K_1 \left[\left(\frac{n}{\beta^n \gamma^{m+1}} \right)^{1/2} + |\nabla E'_P|_m^n - \nabla E'_P|_m^{n-1}| \right] \right] \\ &\leq K(K+1) \left(\frac{n}{\beta^n \gamma^{m+1}} \right)^{1/2} + K(K+1) |\nabla E'_P|_m^n - \nabla E'_P|_m^{n-1}| \\ &\quad + K \|\eta_P|_m^n - \eta_P|_m^{n-1}\|. \end{aligned}$$

Hence, A(i) and A(ii) and $\gamma < \frac{1}{2}$ imply

$$|\nabla E'_P|_{m+1}^n - \nabla E'_P|_{m+1}^{n-1}| \leq K^{3(m+1)+1} \left(\frac{n}{\beta^n \gamma^{m+1}} \right)^{1/2} \left[\left(\frac{1}{K^2} + \frac{1}{K^3} \right) + \left(\frac{1}{K} + \frac{1}{K^2} \right) + \frac{1}{K^2} \right],$$

which by the assumption on K proves C(i). For C(ii), using (155) again, we get

$$\begin{aligned} \|\eta_P|_{m+1}^n - \eta_P|_{m+1}^{n-1}\| &\leq \|\eta_P|_m^n - \eta_P|_m^{n-1}\| + K_1 \left[\left(\frac{n}{\beta^n \gamma^{m+1}} \right)^{1/2} + |\nabla E'_P|_m^n - \nabla E'_P|_m^{n-1}| \right] \\ &\leq K^{3(m+1)+1} \left(\frac{n}{\beta^n \gamma^{m+1}} \right)^{1/2} \left[\frac{1}{K^3} + \frac{1}{K^3} + \frac{1}{K^2} \right], \end{aligned}$$

which by the assumption on K and $\gamma < \frac{1}{2}$ proves C(ii) and concludes the proof. \square

Corollary 7.6. *Let $n > \alpha m \geq 1$. For $|g|$ and γ as in Theorem 7.5 the estimate*

$$\|\phi_P|_m^n - \phi_P|_m^{n-1}\| \leq CmK^{3m+1} \left(\frac{n}{\beta^n \gamma^m} \right)^{1/2}$$

holds true.

Proof. By Definition 7.2 and the unitarity of the transformations W_m we have that

$$\|\phi_P|_m^n - \phi_P|_m^{n-1}\| \leq \| [W_m(\nabla E'_P|_m^n) - W_m(\nabla E'_P|_m^{n-1})] \eta_P|_m^n \| + \|\eta_P|_m^n - \eta_P|_m^{n-1}\|. \quad (176)$$

The lower bound on the norm of $\eta_P|_m^n$ in (154) together with Lemma 6.6 and the constraints (68) yield the estimate

$$\| [W_m(\nabla E'_P|_m^n) - W_m(\nabla E'_P|_m^{n-1})] \eta_P|_m^n \| \leq Cm \left| \nabla E'_P|_m^n - \nabla E'_P|_m^{n-1} \right|.$$

The claim then follows from a direct application of Theorem 7.5. \square

Before we can prove the second main result, we must show the convergence of the fiber Hamiltonians under the simultaneous removal of the UV and IR cut-off, $H_P^{W'}|_m^{n(m)} \rightarrow H_P^{W'}|_\infty$. For this, we need a slightly faster scaling $n(m)$.

Lemma 7.7. *Under the same assumptions of Theorem 7.5, there exist $\bar{\alpha} \geq \alpha$ such that for any $\mathbb{N} \ni \alpha' > \bar{\alpha}$ and $n(m) = \alpha' m$, the Hamiltonians $(H_P^{W'}|_m^{n(m)})_{m \in \mathbb{N}}$ converge in the norm resolvent sense as $m \rightarrow \infty$.*

Proof. The convergence of the resolvent of $H_P^{W'}|_m^{n(m)}$ consists of direct applications of results of Section 4, Section 6 and the present section. Let $z = i\lambda$ with $|\lambda| > 1$. First, we observe that for all $m \in \mathbb{N}$ the range of $(H_P^{W'}|_m^{n(m)} - z)^{-1}$ equals $D(H_{P,0})$ which is dense in \mathcal{F} . By the Trotter-Kato Theorem [RS81, Theorem VIII.22] it suffices to prove that the family of resolvents $([H_P^{W'}|_m^{n(m)} - z]^{-1})_{m \in \mathbb{N}}$ is convergent. We begin with

$$\begin{aligned} \left\| \frac{1}{H_P^{W'}|_m^l - z} - \frac{1}{H_P^{W'}|_m^{l-1} - z} \right\| &\leq \left\| \frac{1}{H_P^{W'}|_m^l - z} - \frac{1}{H_P^{W'}|_m^{l-1} - z} \right\| \\ &+ \left\| \frac{1}{W_m(\nabla E'_P|_m^l)^* H_P^{W'}|_m^{l-1} W_m(\nabla E'_P|_m^l) - z} - \frac{1}{W_m(\nabla E'_P|_m^{l-1})^* H_P^{W'}|_m^{l-1} W_m(\nabla E'_P|_m^{l-1}) - z} \right\| \end{aligned}$$

where we used unitarity of W_m in the first line. Mimicking Corollary 4.7, the first term is bounded above by

$$C|g| \left(\frac{l}{\beta^l} \right)^{1/2}.$$

With the standard inequalities, the second term is bounded by

$$C|g| \left\| \frac{1}{(H_P|_m^{l-1} - z)^{1/2}} \right\| \cdot \left\| (H^f)^{1/2} \frac{1}{(H_P|_m^{l-1} - z)^{1/2}} \right\| \cdot \frac{1}{\tau_m^{1/2}} \left| \nabla E'_P|_m^l - \nabla E'_P|_m^{l-1} \right|,$$

which can be further bounded by

$$C|g| \frac{1}{|\operatorname{Im} z|} \gamma^{-m/2} K^{3m+1} \left(\frac{l}{\beta^l \gamma^m} \right)^{1/2}$$

with the help of Lemma 3.2 and Theorem 7.5. Hence, it holds

$$\left\| \frac{1}{H_P^{W'}|_m^{n(m)} - z} - \frac{1}{H_P^{W'}|_m^{n(m-1)} - z} \right\| \leq \alpha' C |g| K (\alpha' m)^{1/2} \left(\frac{K^3}{\gamma \beta^{\alpha'/2}} \right)^m \quad (177)$$

where

$$\frac{K^3}{\gamma \beta^{\alpha'/2}} < 1$$

for $\alpha' \geq \bar{\alpha}$ and $\bar{\alpha}$ sufficiently large.

Moreover, using the explicit expressions (85), (86), Lemma 3.2, the bound

$$|\nabla E_P'|_m^n - \nabla E_P'|_{m-1}^n| \leq C \gamma^{m/4} \quad (178)$$

at fixed n from Lemma 6.7, and a resolvent expansion one can show that

$$\begin{aligned} & \left\| \frac{1}{H_P^{W'}|_m^{n(m-1)} - z} - \frac{1}{\widetilde{H}_P^{W'}|_m^{n(m-1)} - z} \right\| \\ & \leq \frac{C}{|\operatorname{Im} z|} \left\| \left(\frac{1}{H_P^{W'}|_m^{n(m-1)} - z} \right)^{1/2} [\widetilde{H}_P^{W'}|_m^{n(m-1)} - H_P^{W'}|_m^{n(m-1)}] \left(\frac{1}{H_P^{W'}|_m^{n(m-1)} - z} \right)^{1/2} \right\| \end{aligned} \quad (179)$$

where the right-hand side in (179) can be controlled in terms of (178).

Furthermore, we observe that

$$\left\| \frac{1}{\widetilde{H}_P^{W'}|_m^{n(m-1)} - z} - \frac{1}{H_P^{W'}|_{m-1}^{n(m-1)} - z} \right\| \leq \frac{C |g| \gamma^{(m-1)/2}}{|\operatorname{Im} z|}. \quad (180)$$

by operator estimates similar to those used to control (102).

Finally, for $\alpha' \geq \bar{\alpha}$ and $\bar{\alpha}$ sufficiently large, the estimates in (177), (179) and (180) imply that the family of resolvents $([H_P^{W'}|_m^{n(m)} - z]^{-1})_{m \in \mathbb{N}}$ is a Cauchy sequence in the norm topology, which concludes the proof. \square

We can now prove the second main result, namely the convergence of the ground state vectors $\phi_P|_m^n$ as $n, m \rightarrow \infty$ with $n \equiv n(m)$.

Proof of Theorem 2.2 in Section 2.

(i) Define

$$\alpha_{\min} := \max \left\{ \left| \frac{6 \ln K - \ln |\gamma|}{\ln \beta} \right|, \bar{\alpha} \right\}. \quad (181)$$

For any $\mathbb{N} \ni \alpha' > \alpha_{\min}$, let $n(m) = \alpha' m$. By Theorem 6.3 and Corollary 7.6 we can estimate

$$\begin{aligned} \|\phi_P|_m^{n(m)} - \phi_P|_{m-1}^{n(m-1)}\| &\leq \|\phi_P|_{m-1}^{n(m-1)} - \phi_P|_m^{n(m-1)}\| + \sum_{l=\alpha'(m-1)}^{\alpha'm} \|\phi_P|_m^l - \phi_P|_m^{l-1}\| \\ &\leq m\gamma^{\frac{m-1}{4}} + \alpha' \left[CmK^{3m+1} \left(\frac{\alpha'm}{\beta^{\alpha'(m-1)}\gamma^m} \right)^{1/2} \right] \\ &\leq m\gamma^{\frac{m-1}{4}} + m^{3/2}\alpha'^{3/2}CK\beta^{\alpha'/2} \left(\frac{K^3}{(\beta^{\alpha'}\gamma)^{1/2}} \right)^m \end{aligned}$$

Due to (181) the term $\frac{K^3}{(\beta^{\alpha'}\gamma)^{1/2}} < 1$ so that $(\phi_P|_m^{n(m)})_{m \in \mathbb{N}}$ is a Cauchy sequence. We denote its limit by $\phi_P|_\infty^\infty$. Finally Theorem 6.3 ensures that the vector $\phi_P|_\infty^\infty$ has norm larger than $\frac{1}{2}$.

- (ii) Let $E'_P|_\infty^\infty := \lim_{m \rightarrow \infty} E'_P|_m^\infty$ which exists by Corollary 5.4. By Lemma 7.7 and (i), $E'_P|_\infty^\infty$ is the eigenvalue corresponding to the eigenvector $\phi_P|_\infty^\infty$ of $H_P^{W'}|_\infty^\infty$. Furthermore,

$$\text{Spec}(H_P^{W'}|_m^n) = \text{Spec}(H'_P|_m^n) \subseteq [E'_P|_m^n, \infty).$$

By the nonexpansion property of the norm resolvent convergence for self-adjoint operators [RS81, Theorem VIII.24], this implies that $\phi_P|_\infty^\infty$ is ground state of $H_P^{W'}|_\infty^\infty$ and $E'_P|_\infty^\infty$ is the ground state energy.

□

A Proofs of Lemma 3.2 and Corollary 5.4

Proof of Lemma 3.2. Let $\psi \in D(H_{P,0}^{1/2})$. We start with the identity

$$\langle \psi, H_{P,0}\psi \rangle = \langle \psi, H'_P|_m^n \psi \rangle - \langle \psi, \Delta H'_P|_0^n \psi \rangle - \langle \psi, g\Phi|_m^0 \psi \rangle \quad (182)$$

where

$$\begin{aligned} \langle \psi, \Delta H'_P|_0^n \psi \rangle &= \left\langle \psi, \left[\frac{1}{2} \left((B|_0^n)^2 + (B^*|_0^n)^2 \right) + B^*|_0^n \cdot B|_0^n - (P - P^f) \cdot B|_0^n - B^*|_0^n \cdot (P - P^f) \right] \psi \right\rangle \\ &= \text{Re} \left[\langle \psi, (B|_0^n)^2 \psi \rangle + \langle B|_0^n \psi, B|_0^n \psi \rangle - 2 \langle (P - P^f)\psi, B|_0^n \psi \rangle \right]. \end{aligned}$$

We denote the number operator of bosons in the momentum range $[\kappa, \sigma_n)$ by

$$N|_0^n := \int_{\mathcal{B}_{\sigma_n} \setminus \mathcal{B}_\kappa} dk \, b(k)^* b(k)$$

and express the vector $\psi \in \mathcal{F}$ as a sequence $(\psi^j)_{j \geq 0}$ of j -particle wave functions $\psi^j \in L^2(\mathbb{R}^{3j}, \mathbb{C})$, $j \geq 1$, and $\psi^0 \in \mathbb{C}$. Following [Nel64, Proof of Lemma 5] it is convenient to consider an estimate of the following type

$$\begin{aligned} \text{Re} \langle \psi, (B|_0^n)^2 \psi \rangle &= \text{Re} \langle (N|_0^n + 3)^{1/2} \psi, (N|_0^n + 3)^{-1/2} (B|_0^n)^2 \psi \rangle \\ &\leq \|(N|_0^n + 3)^{1/2} \psi\| \|(N|_0^n + 3)^{-1/2} (B|_0^n)^2 \psi\|. \end{aligned} \quad (183)$$

We consider the two norms in (183) separately. For $I \subset \mathbb{R}_0^+$ let $\mathbb{1}_I(k) \equiv \mathbb{1}_I(|k|)$ denote the characteristic function of I . Schwarz's inequality gives

$$\begin{aligned}
& \left\| (N|_0^n + 3)^{-1/2} (B|_0^n)^2 \psi \right\|^2 \\
& \leq c_1 g^4 \sum_{j=0}^{\infty} \int dk_1 \dots \int dk_{j+2} \frac{(j+1)(j+2) \omega(k_{j+1})^{1/2} \mathbb{1}_{[\kappa, \infty)}(k_{j+1}) \omega(k_{j+2})^{1/2} \mathbb{1}_{[\kappa, \infty)}(k_{j+2})}{\sum_{i=1}^j \mathbb{1}_{[\kappa, \infty)}(k_i) + 3} \times \\
& \quad \times \left| \psi^{(j+2)}(k_1 \dots k_{j+2}) \right|^2 \\
& = c_1 g^4 \sum_{j=0}^{\infty} \int dk_1 \dots \int dk_{j+2} \frac{(j+1)(j+2) \omega(k_{j+1})^{1/2} \omega(k_{j+2})^{1/2} \mathbb{1}_{[\kappa, \infty)}(k_{j+1}) \mathbb{1}_{[\kappa, \infty)}(k_{j+2})}{\sum_{i=1}^{j+2} \mathbb{1}_{[\kappa, \infty)}(k_i) + 1} \times \\
& \quad \times \left| \psi^{(j+2)}(k_1 \dots k_{j+2}) \right|^2 \\
& \leq c_1 g^4 \sum_{j=0}^{\infty} \int dk_1 \dots \int dk_{j+2} (j+1)(j+2) \frac{1}{2} \left[\omega(k_{j+1}) \mathbb{1}_{[\kappa, \infty)}(k_{j+2}) + \omega(k_{j+2}) \mathbb{1}_{[\kappa, \infty)}(k_{j+1}) \right] \times \\
& \quad \times \frac{\left| \psi^{(j+2)}(k_1 \dots k_{j+2}) \right|^2}{\sum_{i=1}^{j+2} \mathbb{1}_{[\kappa, \infty)}(k_i) + 1}. \quad (184)
\end{aligned}$$

for an n -independent and finite constant

$$c_1 := \left(\int dk \left| k \frac{\rho(k)}{\frac{|k|^2}{2} + \omega(k)} \frac{\mathbb{1}_{[\kappa, \infty)}(k)}{\omega(k)^{1/4}} \right|^2 \right)^{1/2}.$$

Using the symmetry we get

$$\begin{aligned}
(184) & = g^4 c_1 \sum_{j=0}^{\infty} \int dk_1 \dots \int dk_{j+2} \sum_{l=1}^{j+2} \sum_{m \neq l} \omega(k_l) \mathbb{1}_{[\kappa, \infty)}(k_m) \frac{\left| \psi^{(j+2)}(k_1 \dots k_{j+2}) \right|^2}{\sum_{i=1}^{j+2} \mathbb{1}_{[\kappa, \infty)}(k_i) + 1} \\
& \leq g^4 c_1 \sum_{j=0}^{\infty} \int dk_1 \dots \int dk_{j+2} \left[\sum_{l=1}^{j+2} \omega(k_l) \right] \frac{\sum_{m=1}^{j+2} \mathbb{1}_{[\kappa, \infty)}(k_m)}{\sum_{i=1}^{j+2} \mathbb{1}_{[\kappa, \infty)}(k_i) + 1} \left| \psi^{(j+2)}(k_1 \dots k_{j+2}) \right|^2 \\
& \leq g^4 c_1 \left\| (H^f)^{1/2} \psi \right\|^2.
\end{aligned}$$

For the remaining term in (183) we compute

$$\langle \psi, (N|_0^n + 3) \psi \rangle \leq \frac{1}{\kappa} \langle \psi, H^f \psi \rangle + 3 \langle \psi, \psi \rangle. \quad (185)$$

Moreover, we estimate

$$\left| \langle \psi, (P - P^f) B|_0^n \psi \rangle \right| \leq \|(P - P^f) \psi\| \|B|_0^n \psi\| \leq \sqrt{2} \|H_{P,0}^{1/2} \psi\| \|B|_0^n \psi\| \quad (186)$$

where by the standard inequalities in (39)

$$\|B|_0^n \psi\| \leq |g| c_2 \|(H^f)^{1/2} \psi\| \quad (187)$$

holds true for an n -independent and finite constant

$$c_2 := \left(\int dk \left| k \frac{\rho(k)}{\frac{|k|^2}{2} + \omega(k)} \frac{\mathbb{1}_{[\kappa, \infty)}(k)}{\omega(k)^{1/2}} \right|^2 \right)^{1/2}.$$

Finally, using the standard inequalities in (22) again, we find

$$\left| \langle \psi, g\Phi_m^0 \psi \rangle \right| \leq 2|g|c_3 \|\psi\| \|(H^f)^{1/2} \psi\| \leq |g|c_3 (\langle \psi, H_{P,0} \psi \rangle + \langle \psi, \psi \rangle) \quad (188)$$

for an m -independent and finite constant

$$c_3 := \left(\int dk \left| \frac{\rho(k) \mathbb{1}_{[0, \kappa)}(k)}{\omega(k)^{1/2}} \right|^2 \right)^{1/2}$$

Hence, for $|g| \leq 1$ the identity (182) and the estimates (183)-(188) yield the bound

$$\left| \langle \psi, \Delta H'_{P,0} \psi \rangle \right| + \left| \langle \psi, g\Phi_m^0 \psi \rangle \right| \leq |g| [c_a \langle \psi, H_{P,0} \psi \rangle + c_b \langle \psi, \psi \rangle] \quad (189)$$

for m and n -independent positive constants c_a and c_b . For $|g| < \frac{1}{c_a}$ inequality (189) proves the claim. \square

Proof of Corollary 5.4.

- (i) We note that $E'_{P,m} \leq \langle \Omega, H'_{P,m} \Omega \rangle = \frac{P^2}{2}$ and, furthermore, by applying Lemma 3.2 we observe that for any $\phi \in D(H_{P,0}^{1/2})$, $\|\phi\| = 1$,

$$0 \leq (1 - |g|c_a) \langle \phi, H_{P,0} \phi \rangle \leq \langle \phi, H'_{P,m} \phi \rangle + |g|c_b.$$

- (ii) First we study the case $|k| < 1$ where we follow a strategy similar to [CFP09, Section VI]:

$$\begin{aligned} E'_{P-k,m} - E'_{P,m} &= \inf_{\|\varphi\|=1} [\langle \varphi, (H'_{P-k,m} - H'_{P,m}) \varphi \rangle + \langle \varphi, H'_{P,m} \varphi \rangle - E'_{P,m}] \\ &\geq \inf_{\|\varphi\|=1} \left[\frac{k^2}{2} - |k| \left| \langle \varphi, (P - P^f + B_0^n + B_0^{*n}) \varphi \rangle \right| + \langle \varphi, H'_{P,m} \varphi \rangle - E'_{P,m} \right] \end{aligned}$$

where the infimum is meant to be taken over $\varphi \in D(H_{P,0}^{1/2}) \cap \mathcal{F}_m^n$ only. By the standard estimates (39) we get

$$\left| \langle \varphi, (P - P^f + B_0^n + B_0^{*n}) \varphi \rangle \right| \leq (\sqrt{2} + 2|g|C) \|H_{P,0}^{1/2} \varphi\| \quad (190)$$

where C does not depend on n since B_0^{*n} can be seen to act to the left as B_0^n and the integral in (39) converges for any $n \in \mathbb{N} \cup \{\infty\}$. Using Lemma 3.2 it turns out that $E'_{P-k,m} - E'_{P,m}$ is bounded from below by

$$\begin{aligned} \inf_{\|\varphi\|=1} \left[\frac{k^2}{2} - |k| \frac{\sqrt{2} + 2C|g|}{\sqrt{1 - |g|c_a}} \sqrt{\langle \varphi, H'_{P,m} \varphi \rangle + |g|c_b + \langle \varphi, H'_{P,m} \varphi \rangle - E'_{P,m}} \right] \\ \geq \inf_{\lambda \geq 0} \left[\frac{k^2}{2} - |k| \frac{\sqrt{2} + 2C|g|}{\sqrt{1 - |g|c_a}} \sqrt{\lambda + E'_{P,m} + |g|c_b + \lambda} \right] =: \inf_{\lambda \geq 0} f(\lambda) \end{aligned}$$

where

$$f(\lambda) := \frac{k^2}{2} - |k| \frac{\sqrt{2} + 2C|g|}{\sqrt{1 - |g|c_a}} \sqrt{\lambda + E'_P|_m^n + |g|c_b} + \lambda \quad (191)$$

The infimum can be attained either at $\lambda^* = 0$ or at λ^* such that $f'(\lambda^*) = 0$, i.e.

$$\lambda^* = \frac{|k|^2 (\sqrt{2} + 2C|g|)^2}{4(1 - |g|c_a)} - (E'_P|_m^n + |g|c_b) \quad (192)$$

Case $\lambda^* = 0$: Since

$$f(0) \geq -|k| \frac{\sqrt{2} + 2C|g|}{\sqrt{1 - |g|c_a}} \sqrt{E'_P|_m^n + |g|c_b}$$

and, by claim (ii),

$$0 \leq E'_P|_m^n + |g|c_b \leq \frac{P^2}{2} + |g|c_b \leq \frac{P_{\max}^2}{2} + |g|c_b,$$

we obtain the lower bound

$$f(0) \geq -|k| \frac{\sqrt{2} + 2C|g|}{\sqrt{1 - |g|c_a}} \left(\frac{P_{\max}}{\sqrt{2}} + O(|g|) \right) = -|k|P_{\max} (1 + O(|g|)). \quad (193)$$

Case $\lambda^* > 0$: To evaluate

$$f(\lambda^*) = \frac{k^2}{2} \left(1 - \frac{1}{2} \frac{(\sqrt{2} + 2C|g|)^2}{1 - |g|c_a} \right) - (E'_P|_m^n + |g|c_b)$$

we consider that λ^* given in (192) is assumed to be larger than zero. This implies that

$$f(\lambda^*) > \frac{k^2}{2} \left(1 - \frac{(\sqrt{2} + 2C|g|)^2}{1 - |g|c_a} \right) = -k^2 \left(\frac{1}{2} + O(g) \right) > -|k| \left(\frac{1}{2} + O(g) \right) \quad (194)$$

where we have used that $|k| < 1$.

Recall that $P_{\max} = \frac{1}{4}$. Therefore, taking the minimum of both lower bounds (193) and (194) for $|g|$ sufficiently small proves that, for all $|k| < 1$,

$$E'_{P-k}|_m^n - E'_P|_m^n \geq -c|k|, \quad (195)$$

for any $c > \frac{1}{2}$, and in particular for $c = C_{\nabla E} := \frac{3}{4}$.

For the case $|k| \geq 1$ Theorem 3.1 implies:

$$E'_{P-k}|_m^n - E'_P|_m^n = (E'_{P-k}|_m^n - E'_0|_m^n) + (E'_0|_m^n - E'_P|_m^n) \geq E'_0|_m^n - E'_P|_m^n \quad (196)$$

$$\geq -C_{\nabla E}|P_{\max}| \geq -C_{\nabla E}|k|, \quad (197)$$

where the step from (196) to (197) is justified by invoking the result in the case $|k| < 1$, i.e., by replacing $k = P$ in (195).

(iii) Let $\Psi'_{p|m}{}^n$ be the eigenvector corresponding to $E'_{p|m}{}^n$, then we get

$$E'_{p|m}{}^{n+1} \leq \left\langle \frac{\Psi'_{p|m}{}^n}{\|\Psi'_{p|m}{}^n\|} \otimes \Omega, [H'_{p|0}{}^n + \Delta H'_{p|n}{}^{n+1} + g\Phi_m^0] \frac{\Psi'_{p|m}{}^n}{\|\Psi'_{p|m}{}^n\|} \otimes \Omega \right\rangle = \left\langle \frac{\Psi'_{p|m}{}^n}{\|\Psi'_{p|m}{}^n\|}, H'_{p|0}{}^n \frac{\Psi'_{p|m}{}^n}{\|\Psi'_{p|m}{}^n\|} \right\rangle = E'_{p|m}{}^n$$

as well as

$$E'_{p|m+1}{}^n \leq \left\langle \frac{\Psi'_{p|m}{}^n}{\|\Psi'_{p|m}{}^n\|} \otimes \Omega, [H'_{p|m}{}^n + g\Phi_{m+1}^n] \frac{\Psi'_{p|m}{}^n}{\|\Psi'_{p|m}{}^n\|} \otimes \Omega \right\rangle = \left\langle \frac{\Psi'_{p|m}{}^n}{\|\Psi'_{p|m}{}^n\|}, H'_{p|m}{}^n \frac{\Psi'_{p|m}{}^n}{\|\Psi'_{p|m}{}^n\|} \right\rangle = E'_{p|m}{}^n.$$

□

B Transformed Hamiltonians: derivation of identities (85), (86) and (92)

Derivation of identity (85). Let $n, m \in \mathbb{N}$. Recalling (6) we can start with the expression

$$\begin{aligned} H'_{p|m}{}^n &= \frac{1}{2} (P - P^f)^2 + H^f + \frac{1}{2} [(B|_0^n)^2 + (B^*|_0^n)^2] + B^*|_0^n \cdot B|_0^n \\ &\quad - (P - P^f) \cdot B|_0^n - B^*|_0^n \cdot (P - P^f) + g\Phi_m^0. \end{aligned}$$

This Hamiltonian can be written in the form

$$H'_{p|m}{}^n = \frac{1}{2} (P - P^f - B|_0^n - B^*|_0^n)^2 + H^f + g\Phi_m^0 + S_{P,n}$$

where we collected terms acting in the ultraviolet region in

$$S_{P,n} := -\frac{1}{2} ([B|_0^n, P - P^f] + [P - P^f, B^*|_0^n] + [B|_0^n, B^*|_0^n]).$$

The conjugation by $W_m(\nabla E'_{p|m}{}^n)$ on these various terms reads

$$\begin{aligned} W_m(\nabla E'_{p|m}{}^n) P^f W_m(\nabla E'_{p|m}{}^n)^* &= P^f + A_{P,m}^{(n)} + C_{P,m}^{(k,n)} \\ W_m(\nabla E'_{p|m}{}^n) H^f W_m(\nabla E'_{p|m}{}^n)^* &= H^f + L_{P,m}^{(n)} + C_{P,m}^{(\omega,n)} \\ W_m(\nabla E'_{p|m}{}^n) \Phi_m^0 W_m(\nabla E'_{p|m}{}^n)^* &= \Phi_m^0 + C_{P,m}^{(\rho,n)} \\ W_m(\nabla E'_{p|m}{}^n) S_{P,n} W_m(\nabla E'_{p|m}{}^n)^* &= S_{P,n} \end{aligned} \tag{198}$$

for

$$L_{P,m}^{(n)} := \int dk \, \omega(k) \, \alpha_m(\nabla E'_{p|m}{}^n, k) [b(k) + b^*(k)].$$

and $A_{P,m}^{(n)}, C_{P,m}^{(k,n)}, C_{P,m}^{(\omega,n)}, C_{P,m}^{(\rho,n)}$ given in equations (82).

Using these formulae we find

$$\begin{aligned} W_m(\nabla E'_{p|m}{}^n) H'_{p|m}{}^n W_m(\nabla E'_{p|m}{}^n)^* &= \frac{1}{2} (P - P^f - A_{P,m}^{(n)} - B|_0^n - B^*|_0^n - C_{P,m}^{(k,n)})^2 \\ &\quad + (H^f + L_{P,m}^{(n)} + C_{P,m}^{(\omega,n)}) + (g\Phi_m^0 + C_{P,m}^{(\rho,n)}) + S_{P,n}. \end{aligned}$$

Applying the identity (79) we further have

$$\begin{aligned} P &= \nabla E'_P|_m^n + \left\langle [P^f + B|_0^n + B^*|_0^n] \right\rangle_{\Psi'_P|_m^n} = \nabla E'_P|_m^n + \left\langle P^f + A_{P,m}^{(n)} + B|_0^n + B^*|_0^n \right\rangle_{W_m \Psi'_P|_m^n} + C_{P,m}^{(k,n)} \\ &= \nabla E'_P|_m^n + \langle \Pi_P|_m^n \rangle_{\phi_P|_m^n} + C_{P,m}^{(k,n)}, \end{aligned} \quad (199)$$

so that we obtain

$$\begin{aligned} &W_m(\nabla E'_P|_m^n) H'_P|_m^n W_m(\nabla E'_P|_m^n)^* \\ &= \frac{1}{2} \left(\nabla E'_P|_m^n + \left\langle P^f + A_{P,m}^{(n)} + B|_0^n + B^*|_0^n \right\rangle_{\phi_P|_m^n} - \left(P^f + A_{P,m}^{(n)} + B|_0^n + B^*|_0^n \right) \right)^2 \\ &\quad + H^f + L_{P,m}^{(n)} + C_{P,m}^{(\omega,n)} + g\Phi|_{\tau_m}^\kappa + C_m^{(\rho)} + S_{P,n} \\ &= \frac{1}{2} \Gamma_P|_m^n{}^2 + \frac{1}{2} \nabla E'_P|_m^n{}^2 \\ &\quad + \nabla E'_P|_m^n \cdot \left(\left\langle P^f + A_{P,m}^{(n)} + B|_0^n + B^*|_0^n \right\rangle_{\phi_P|_m^n} - \left(P^f + A_{P,m}^{(n)} + B|_0^n + B^*|_0^n \right) \right) \\ &\quad + H^f + L_{P,m}^{(n)} + C_{P,m}^{(\omega,n)} + g\Phi|_{\tau_m}^\kappa + C_{P,m}^{(\rho,n)} + S_{P,n}. \end{aligned}$$

The transformation W_m was designed to yield the following cancellation

$$-\nabla E'_P|_m^n \cdot A_{P,m}^{(n)} + L_m + g\Phi|_{\tau_m}^\kappa = 0. \quad (200)$$

Hence, using the abbreviations introduced in the beginning of Section 6, we finally arrive at the form

$$H_P^{W'}|_m^n := W_m(\nabla E'_P|_m^n) H'_P|_m^n W_m(\nabla E'_P|_m^n)^* = \frac{1}{2} \Gamma_P|_m^n{}^2 + H^f - \nabla E'_P|_m^n \cdot P^f + C_{P,m}^{(n)} + R_P|_m^n. \quad (201)$$

By analogous methods as in [Nel64] for the ultraviolet region it can then be verified that this equality actually holds on $D(H_{P,0})$. \square

Derivation of Identity (86). From the definition of $\widetilde{H}_P^{W'}|_m^n$, we can write

$$\widetilde{H}_P^{W'}|_m^n = W_m(\nabla E'_P|_{m-1}^n) W_{m-1}(\nabla E'_P|_{m-1}^n)^* [H_P^{W'}|_{m-1}^n + g\Phi|_m^{m-1}] W_{m-1}(\nabla E'_P|_{m-1}^n) W_m(\nabla E'_P|_{m-1}^n)^*$$

which by virtue of the formulae (198) as well as identity (201) gives

$$\begin{aligned} \widetilde{H}_P^{W'}|_m^n &= \frac{1}{2} \left(\Gamma_P|_{m-1}^n + \widetilde{A}_{P,m}^{(n)} - A_{P,m-1}^{(n)} + \widetilde{C}_{P,m}^{(k,n)} - C_{P,m-1}^{(k,n)} \right)^2 \\ &\quad + H^f + \widetilde{L}_{P,m}^{(n)} - L_{P,m-1} + \widetilde{C}_{P,m}^{(\omega,n)} - C_{P,m-1}^{(\omega,n)} \\ &\quad - \nabla E'_P|_{m-1}^n \cdot \left(P^f + \widetilde{A}_{P,m}^{(n)} - A_{P,m-1}^{(n)} + \widetilde{C}_{P,m}^{(k,n)} - C_{P,m-1}^{(k,n)} \right) \\ &\quad + g\Phi|_m^0 + \widetilde{C}_{P,m}^{(\rho,n)} - C_{P,m-1}^{(\rho,n)} + C_{P,m-1}^{(n)} + R_P|_{m-1}^n \end{aligned}$$

for

$$\widetilde{L}_{P,m}^{(n)} := \int dk \, \omega(k) \, \alpha_m(\nabla E'_P|_{m-1}^n, k) [b(k) + b^*(k)].$$

Due to the cancellation (200) and

$$\widetilde{C}_{P,m}^{(k,n)} = C_{P,m-1}^{(k,n)} - \nabla E'_{P,m-1} \cdot (\widetilde{C}_{P,m}^{(k,n)} - C_{P,m-1}^{(k,n)}) + \widetilde{C}_{P,m}^{(\omega,n)} - C_{P,m-1}^{(\omega,n)} + \widetilde{C}_{P,m}^{(\rho,n)} - C_{P,m-1}^{(\rho,n)}$$

we finally obtain

$$\widetilde{H}_P^{W'}|_m^n = \frac{1}{2}(\Gamma_P|_{m-1}^n + \widetilde{A}_{P,m}^{(n)} - A_{P,m-1}^{(n)} + \widetilde{C}_{P,m}^{(k,n)} - C_{P,m-1}^{(k,n)})^2 + H^f - \nabla E'_{P,m-1} \cdot P^f + \widetilde{C}_{P,m}^{(n)} + R_P|_{m-1}^n.$$

One can verify that this identity holds on $D(H_{P,0})$. \square

Derivation of Identity (92). By definitions (84) and (88),

$$\widetilde{\Gamma}_P|_m^n - \Gamma_P|_{m-1}^n = \langle \Pi_P|_m^n \rangle_{\phi_P|_{m-1}^n} - \langle \widetilde{\Pi}_P|_m^n \rangle_{\widetilde{\phi}_P|_m^n} + \widetilde{\Pi}_P|_m^n - \Pi_P|_m^n.$$

so that (199) yields

$$\widetilde{\Gamma}_P|_m^n - \Gamma_P|_{m-1}^n = \nabla E'_{P,m} - \nabla E'_{P,m-1} + \widetilde{A}_{P,m}^{(n)} - A_{P,m-1}^{(n)} + \widetilde{C}_{P,m}^{(k,n)} - C_{P,m-1}^{(k,n)}.$$

One can verify that this identity holds on $D(H_{P,0})$. \square

References

- [AH10] Abdelmalek Abdesselam and David Hasler. Analyticity of the ground state energy for massless nelson models. *1008.4628*, August 2010.
- [BCFS07] Volker Bach, Thomas Chen, Jürg Fröhlich, and Israel Michael Sigal. The renormalized electron mass in non-relativistic quantum electrodynamics. *Journal of Functional Analysis*, 243(2):426–535, February 2007.
- [Can71] John T. Cannon. Quantum field theoretic properties of a model of nelson: Domain and eigenvector stability for perturbed linear operators. *Journal of Functional Analysis*, 8(1):101–152, August 1971.
- [CFP09] Thomas Chen, Jürg Fröhlich, and Alessandro Pizzo. Infraparticle scattering states in nonrelativistic quantum electrodynamics. II. mass shell properties. *Journal of Mathematical Physics*, 50(1):012103, 2009.
- [Che08] Thomas Chen. Infrared renormalization in non-relativistic QED and scaling criticality. *Journal of Functional Analysis*, 254(10):2555–2647, May 2008.
- [Frö73] Jürg Fröhlich. On the infrared problem in a model of scalar electrons and massless, scalar bosons. *Ann. Inst. H. Poincaré Sect. A (N.S.)*, 19:1–103, 1973.
- [Gro62] Eugene P. Gross. Particle-like solutions in field theory. *Annals of Physics*, 19(2):219–233, August 1962.
- [Gro72] Leonard Gross. Existence and uniqueness of physical ground states. *Journal of Functional Analysis*, 10(1):52–109, May 1972.

- [HHS05] Masao Hirokawa, Fumio Hiroshima, and Herbert Spohn. Ground state for point particles interacting through a massless scalar bose field. *Advances in Mathematics*, 191(2):339–392, March 2005.
- [Kat95] Tosio Kato. *Perturbation Theory for Linear Operators*. Springer, February 1995.
- [Nel64] Edward Nelson. Interaction of nonrelativistic particles with a quantized scalar field. *Journal of Mathematical Physics*, 5(9):1190, 1964.
- [Piz03] Alessandro Pizzo. One-particle (improper) states in nelson’s massless model. *Annales Henri Poincare*, 4(3):439–486, June 2003.
- [RS81] Michael Reed and Barry Simon. *Methods of Modern Mathematical Physics, Vol. 1: Functional Analysis*. Academic Pr Inc, REV AND ENL. edition, February 1981.